PDE and Boundary-Value Problems Winter Term 2016/2017

Lecture 8

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Purpose of Lesson

- To define the exponential Fourier and inverse exponential Fourier transforms, to illustrate several useful properties of the Fourier transform and to show how these properties can be used to solve PDEs.
- To define the Laplace and inverse Laplace transforms, to illustrate several useful properties of the Laplace transform and to show how these properties can be used to solve PDEs.

Exponential Fourier transforms:

$$\mathcal{F}[f] \equiv F(\xi) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \left[f(x) e^{-i\xi x} dx \right] \quad \text{(Fourier transform - FT)}$$

$$\mathcal{F}^{-1}[F] \equiv f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \left[F(\xi) e^{i\xi x} d\xi \right] \quad \text{(inverse FT)}$$

Exponential Fourier transforms:

Remarks

- The Fourier transform $F(\xi)$ can be a complex function; for example, the Fourier transform of

$$f(x) = \begin{cases} 0, & x \leqslant 0 \\ e^{-x}, & x > 0 \end{cases}$$

is $F(\xi) = \dfrac{1}{\sqrt{2\pi}} \dfrac{1 - i\xi}{1 + \xi^2}$.

- Not all functions have Fourier transforms; in fact, $f(x) = c$, e^x, x^2, do not have Fourier transforms. Only functions that go to zero sufficiently fast as $|x| \to \infty$ have transforms.

Properties of the Fourier Transform:

- Transformation of partial derivatives:

$$\mathcal{F}[u_x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_x(x,t) e^{-i\xi x} dx = i\xi \mathcal{F}[u]$$

$$\mathcal{F}[u_{xx}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{xx}(x,t) e^{-i\xi x} dx = -\xi^2 \mathcal{F}[u]$$

$$\mathcal{F}[u_t] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t(x,t) e^{-i\xi x} dx = \frac{\partial}{\partial t} \mathcal{F}[u]$$

$$\mathcal{F}[u_{tt}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{tt}(x,t) e^{-i\xi x} dx = \frac{\partial^2}{\partial t^2} \mathcal{F}[u]$$

Properties of the Fourier Transform:

- The Fourier transform is a linear transformation; that is

$$\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g]$$

- The transform of a product of two functions $f(x) \cdot g(x)$ is not the product of the individual transforms; that is,

$$\mathcal{F}[f(x) \cdot g(x)] \neq \mathcal{F}[f] \cdot \mathcal{F}[g]$$

- Convolution property:

$$\mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g],$$

  where

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x - \xi)g(\xi)d\xi.$$

Properties of the Fourier Transform:

- From the convolution property it follows that

$$f * g = \mathcal{F}^{-1}\left(\mathcal{F}[f] \cdot \mathcal{F}[g]\right).$$

Remark

Hence, to find

$$\mathcal{F}^{-1}\left(\mathcal{F}[f] \cdot \mathcal{F}[g]\right),$$

all we have to do is find the inverse transform of each factor to get f and g and then compute their convolution.

Exponential Fourier transform

	$f(x) = \mathcal{F}^{-1}[F]$	$F(\omega) = \mathcal{F}[f]$
1.	$f^{(n)}(x)$ (n^{th} derivative)	$(i\omega)^n F(\omega)$
2.	$f(ax),\ a > 0$	$\frac{1}{a} F\left(\frac{\omega}{a}\right)$
3.	$f(x-a)$	$e^{-ia\omega} F(\omega)$
4.	$e^{-a^2x^2}$	$\frac{1}{a\sqrt{2}} e^{-\omega^2/(4a^2)}$
5.	$e^{-a\|x\|}$	$\sqrt{\frac{2}{\pi}} \frac{a}{a^2+\omega^2}$

Exponential Fourier transform (cont.)

	$f(x) = \mathcal{F}^{-1}[F]$	$F(\omega) = \mathcal{F}[f]$
6.	$\begin{cases} 1, & \lvert x\rvert < a \\ 0, & \lvert x\rvert > a \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(a\omega)}{\omega}$
7.	$\delta(x-a)$	$\frac{1}{\sqrt{2\pi}}e^{-ia\omega}$
8.	$(1+x^2)^{-1}$	$\sqrt{\frac{\pi}{2}}e^{-\lvert\omega\rvert}$
9.	$xe^{-a\lvert x\rvert},\ a>0$	$-2\sqrt{\frac{2}{\pi}}\frac{ia\omega}{(\omega^2+a^2)^2}$

Exponential Fourier transform (cont.)

	$f(x) = \mathcal{F}^{-1}[F]$	$F(\omega) = \mathcal{F}[f]$
10.	$H(x+a) - H(x-a)$	$\sqrt{\frac{2}{\pi}} \frac{\sin(a\omega)}{\omega}$
11.	$\frac{a}{x^2 + a^2}$	$\sqrt{\frac{\pi}{2}} e^{-a\lvert\omega\rvert}$
12.	$\frac{2ax}{(x^2 + a^2)^2}$	$-i\sqrt{\frac{\pi}{2}} \omega e^{-a\lvert\omega\rvert}$
13.	$\begin{cases} \cos(ax), & \lvert x\rvert < \pi/(2a) \\ 0, & \lvert x\rvert > \pi/(2a) \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{a}{a^2 - \omega^2} \cos(\pi\omega/(2a))$

Exponential Fourier transform (cont.)

	$f(x) = \mathcal{F}^{-1}[F]$	$F(\omega) = \mathcal{F}[f]$
14.	$\begin{cases} 1-\lvert x\rvert, & \lvert x\rvert < 1 \\ 0, & \lvert x\rvert > 1 \end{cases}$	$2\sqrt{\frac{2}{\pi}}\left[\frac{\sin(\omega/2)}{\omega}\right]^2$
15.	$\cos(ax)$	$\sqrt{\frac{\pi}{2}}\,[\delta(\omega+a)+\delta(\omega-a)]$
16.	$\sin(ax)$	$i\sqrt{\frac{\pi}{2}}\,[\delta(\omega+a)+\delta(\omega-a)]$

Solution an Initial-Value Problem

Consider the heat flow in an infinite rod where the initial temperature is $u(x,0) = \phi(x)$. In other words, we look for the solution to the initial-value problem (IVP), sometimes called a Cauchy problem.

Problem 8-1

To find the function $u(x,t)$ that satisfies

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \qquad -\infty < x < \infty, \quad 0 < t < \infty$$

$$\text{IC:} \quad u(x,0) = \phi(x), \qquad -\infty < x < \infty$$

There are three basic steps in solving this problem.

Step 1. (Transformation)

- Since the space variable x ranges from $-\infty$ to ∞, we take the Fourier transform of the PDE and IC with respect to x.

$$\begin{aligned}
\mathcal{F}[u(x,t)] &=: U(\xi,t) = U(t) \\
\mathcal{F}[u_t(x,t)] &= \frac{\partial}{\partial t}\mathcal{F}[u(x,t)] = \frac{\partial}{\partial t}U(t) \\
\mathcal{F}[u_{xx}(x,t)] &= -\xi^2\mathcal{F}[u(x,t)] = \xi^2 U(t) \\
\mathcal{F}[u(x,0)] &= U(0) \\
\mathcal{F}[\phi(x)] &=: \Phi(\xi) \quad (\Phi \text{ is the Fourier transform of } \phi)
\end{aligned}$$

- Substituting all these terms into problem 8-1, we get

$$\begin{aligned}
\text{ODE:} &\quad \frac{d}{dt}U(t) = -\alpha^2\xi^2 U(t) \\
\text{IC:} &\quad U(0) = \Phi(\xi)
\end{aligned} \tag{8.1}$$

Step 2. (Solving the transformed problem)

- Remember the new variable ξ is nothing more than a constant in this differential equation, so the solution to problem (8.1) is

$$U(t) = \Phi(\xi)e^{-\alpha^2\xi^2 t}.$$

Step 3. (Finding the inverse transform)

- We merely compute

$$u(x,t) = \mathcal{F}^{-1}\left[U(\xi,t)\right] = \mathcal{F}^{-1}\left[\Phi(\xi)e^{-\alpha^2\xi^2 t}\right]$$

Step 3. (Finding the inverse transform)

- Due to the convolution property we can write

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left[\Phi(\xi)e^{-\alpha^2\xi^2 t}\right] \\ &= \mathcal{F}^{-1}\left[\Phi(\xi)\right] * \mathcal{F}^{-1}\left[e^{-\alpha^2\xi^2 t}\right] \\ &= \phi(x) * \left[\frac{1}{\alpha\sqrt{2t}}e^{-x^2/(4\alpha^2 t)}\right] \\ &= \frac{1}{2\alpha\sqrt{\pi t}}\int\limits_{-\infty}^{\infty}\phi(\xi)e^{-(x-\xi)^2/(4\alpha^2 t)}d\xi \end{aligned}$$

- Therefore,

$$\boxed{u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int\limits_{-\infty}^{\infty}\phi(\xi)e^{-(x-\xi)^2/(4\alpha^2 t)}d\xi}.$$

Remarks

- The major drawback of the Fourier transform is that all functions can not be transformed;
- Only functions that damp to zero sufficiently fast as $|x| \to \infty$ have transforms.
- The Fourier transform could not be used to transform the time variable in problem 8-1, since $0 < t < \infty$.

Laplace transforms:

$$\mathcal{L}[f] = F(s) = \int\limits_0^\infty f(t)e^{-st}dt \quad \text{(Laplace transform)}$$

$$\mathcal{L}^{-1}[F] = f(t) = \int\limits_{c-i\infty}^{c+i\infty} F(s)e^{st}ds \quad \text{(inverse Laplace transform)}$$

Remarks

- The Laplace transform has one major advantage over the Fourier transform in that the damping factor e^{-st} in the integrand allows us to transform a wider class of functions.
- The factor $e^{i\xi x}$ in the Fourier transform doesn't do any damping, since its absolute value is one.

Sufficient Conditions to Insure the Existence of a Laplace Transform

If

1. f is piecewise continuous on the intervall $0 \leqslant t \leqslant A$ for any positive A;
2. we can find constants M and a such that $|f(t)| \leqslant Me^{at}$ for all values of t greater than some number T

then the Laplace transform

$$\mathcal{L}[f] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

exists for $s > a$.

Properties of the Laplace transform

- Transformation of partial derivatives

$$\mathcal{L}[u_t] = \int_0^\infty u_t(x,t)e^{-st}dt = sU(x,s) - u(x,0)$$

$$\mathcal{L}[u_{tt}] = \int_0^\infty u_{tt}(x,t)e^{-st}dt = s^2U(x,s) - su(x,0) - u_t(x,0)$$

$$\mathcal{L}[u_x] = \int_0^\infty u_x(x,t)e^{-st}dt = \frac{\partial}{\partial x}U(x,s)$$

$$\mathcal{L}[u_{xx}] = \int_0^\infty u_{xx}(x,t)e^{-st}dt = \frac{\partial^2}{\partial x^2}U(x,s)$$

Properties of the Laplace transform

- Convolution property:

$$\mathcal{L}[f * g] = \mathcal{L}[f] \cdot \mathcal{L}[g],$$

where

$$(f * g)(t) = \int\limits_0^t f(\tau)g(t-\tau)d\tau = \int\limits_0^t f(t-\tau)g(\tau)d\tau$$

is the finite convolution of two functions f and g.

- It is evident that

$$\mathcal{L}^{-1}\left\{\mathcal{L}[f] \cdot \mathcal{L}[g]\right\} = f * g$$

Laplace transform

	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
1.	1	$\dfrac{1}{s}, \quad s > 0$
2.	e^{at}	$\dfrac{1}{s-a}, \quad s > a$
3.	$\sin(at)$	$\dfrac{a}{s^2+a^2}, \quad s > 0$
4.	$\cos(at)$	$\dfrac{s}{s^2+a^2}, \quad s > 0$
5.	t^n ($n =$ positive integer)	$\dfrac{n!}{s^{n+1}}, \quad s > 0$

Laplace transform (cont.)

	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
6.	$\sinh(at)$	$\dfrac{a}{s^2 - a^2}, \quad s > \lvert a \rvert$
7.	$\cosh(at)$	$\dfrac{s}{s^2 - a^2}, \quad s > \lvert a \rvert$
8.	$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2 + b^2}, \quad s > a$
9.	$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2 + b^2}, \quad s > a$
10.	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$

Laplace transform (cont.)

	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
11.	$H(t-a)$	$\dfrac{e^{-as}}{s}, \quad s>0$
12.	$H(t-a)f(t-a)$	$e^{-as}F(s)$
13.	$e^{at}f(t)$	$F(s-a)$
14.	$f^{(n)}(t)$ (nth derivative)	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
15.	$f(at)$	$\dfrac{1}{a}F\left(\dfrac{s}{a}\right), \quad a>0$

Laplace transform (cont.)

	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
16.	$\int\limits_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
17.	$\operatorname{erf}(t/2a)$	$\frac{1}{s}e^{a^2s^2}\operatorname{erfc}(as)$
18.	$\operatorname{erfc}(a/(2\sqrt{t}))$	$\frac{1}{s}e^{-a\sqrt{s}}$
19.	$\delta(t-a)$	e^{-sa}
20.	$\frac{1}{\sqrt{\pi t}} - ae^{a^2t}\operatorname{erfc}(a\sqrt{t})$	$\frac{1}{\sqrt{s}+a}$

Laplace transform (cont.)

	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
21.	$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$
22.	$\dfrac{a}{2\sqrt{\pi t^3}} \exp\left(-\dfrac{a^2}{4t}\right)$	$e^{-a\sqrt{s}}$
23.	$\dfrac{1}{\sqrt{\pi t}} \exp\left(-\dfrac{a^2}{4t}\right)$	$\dfrac{1}{\sqrt{s}} e^{-a\sqrt{s}}$
24.	$-e^{ab+a^2 t} \operatorname{erfc}\left(a\sqrt{t} + \dfrac{b}{2\sqrt{t}}\right) + \operatorname{erfc}\left(\dfrac{b}{2\sqrt{t}}\right)$	$\dfrac{ae^{-b\sqrt{s}}}{s(\sqrt{s} + a)}$

Heat Conduction in a Semi Infinite Medium

Consider a deep container of liquid that is insulated on the sides. Suppose the liquid has an initial temperature of u_0 and the temperature of the air above the liquid is zero (some reference temperature). Our goal is to find the temperature of the liquid at various depths of the container at different values of time.

Problem 8-2

To find the function $u(x, t)$ that satisfies

PDE: $u_t = u_{xx}, \quad 0 < x < \infty, \quad 0 < t < \infty$

BC: $u_x(0, t) - u(0, t) = 0, \quad 0 < t < \infty$

IC: $u(x, 0) = u_0, \quad 0 < x < \infty$

Step 1. (Transformation)

- We take the Laplace transform with respect to t-variable. We transform the PDE and the BC - not the IC! As a result we get and ODE in x

$$\begin{aligned} &\text{ODE:} \quad sU(x) - u_0 = \frac{d^2}{dx^2}U(x), \qquad 0 < x < \infty \\ &\text{BC:} \quad \frac{d}{dx}U(0) = U(0) \end{aligned} \tag{8.2}$$

Step 2. (Solving the BVP for ODE)

- The first equation in (8.2) is a second-order ODE with one BC at $x = 0$.
- For physical reasons, we really have a second, implied BC that says $U(x)$ is bounded.
- To solve (8.2), we first find the general solution (homogeneous + a particular solution), which is

$$U(x) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{u_0}{s}$$

- Note that $c_1 = 0$ or else the temperature will go to infinity as x gets large.
- Finding c_2 from the BC at $x = 0$ provides

$$U(x) = -u_0 \left\{ \frac{e^{-\sqrt{s}x}}{s(\sqrt{s}+1)} \right\} + \frac{u_0}{s}$$

Step 3. (Inverse transform)

- The last step is to find the inverse transform of $U(s)$; that is,

$$u(x,t) = \mathcal{L}^{-1}\left[U(x,s)\right].$$

Using the tables we see that

$$\boxed{u(x,t) = u_0 - u_0\left[\operatorname{erfc}\left(x/(2\sqrt{t})\right) - \operatorname{erfc}\left(\sqrt{t} + x/(2\sqrt{t})\right)e^{x+t}\right]}.$$