UNIVERSITÄT DES SAARLANDES
FR 6.1 Mathematik
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Lösung zur 4. Übung zur Vorlesung

Höhere Mathematik für Ingenieure IV

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1. Aufgabe

1.

$$\overline{\partial} f = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) = 0$$

$$\Leftrightarrow \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \quad \text{und} \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$\Leftrightarrow$ Cauchy-Riemann-Differentialgleichungen

$\Leftrightarrow$ f komplex differenzierbar.

2.

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} \overset{\text{C-R-DGlen}}{=} \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial x \partial y},\\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \overset{\text{C-R-DGlen}}{=} \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial y \partial x},\end{aligned}$$

Nach dem Satz von Schwarz lassen sich partielle Ableitungen von C^2-Funktionen vertauschen, also gilt

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \quad \Rightarrow \quad \Delta u = 0.$$

Analog:

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \quad , \quad \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y} \quad \Rightarrow \quad \Delta v = 0.$$

2. Aufgabe

$$u(x,y) + iv(x,y) = f(re^{i\varphi}) = g(r,\varphi) + ih(r,\varphi)$$

mit

$$\begin{aligned} r(x,y) &= \sqrt{x^2+y^2},\\ \varphi(x,y) &= \arctan\left(\frac{y}{x}\right) \quad (\pm\pi). \end{aligned}$$

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{1}{2\sqrt{x^2+y^2}}2x = \frac{x}{r} = \cos(\varphi), \\
\frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} = \sin(\varphi), \\
\frac{\partial \varphi}{\partial x} &= \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{\sin(\varphi)}{r}, \\
\frac{\partial \varphi}{\partial y} &= \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{\cos(\varphi)}{r}, \\
\frac{\partial u}{\partial x} &= \frac{\partial g}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial g}{\partial \varphi}\frac{\partial \varphi}{\partial x} = \frac{\partial g}{\partial r}\cos(\varphi) - \frac{\partial g}{\partial \varphi}\frac{\sin(\varphi)}{r}, \\
\frac{\partial u}{\partial y} &= \frac{\partial g}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial g}{\partial \varphi}\frac{\partial \varphi}{\partial y} = \frac{\partial g}{\partial r}\sin(\varphi) + \frac{\partial g}{\partial \varphi}\frac{\cos(\varphi)}{r}, \\
\frac{\partial v}{\partial x} &= \frac{\partial h}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial h}{\partial \varphi}\frac{\partial \varphi}{\partial x} = \frac{\partial h}{\partial r}\cos(\varphi) - \frac{\partial h}{\partial \varphi}\frac{\sin(\varphi)}{r}, \\
\frac{\partial v}{\partial y} &= \frac{\partial h}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial h}{\partial \varphi}\frac{\partial \varphi}{\partial y} = \frac{\partial h}{\partial r}\sin(\varphi) + \frac{\partial h}{\partial \varphi}\frac{\cos(\varphi)}{r}
\end{aligned}$$

Damit sind die Cauchy-Riemann-Differentialgleichungen

$$\begin{aligned}
\frac{\partial g}{\partial r}\cos(\varphi) - \frac{\partial g}{\partial \varphi}\frac{\sin(\varphi)}{r} &= \frac{\partial h}{\partial r}\sin(\varphi) + \frac{\partial h}{\partial \varphi}\frac{\cos(\varphi)}{r}, \\
\frac{\partial g}{\partial r}\sin(\varphi) + \frac{\partial g}{\partial \varphi}\frac{\cos(\varphi)}{r} &= -\frac{\partial h}{\partial r}\cos(\varphi) + \frac{\partial h}{\partial \varphi}\frac{\sin(\varphi)}{r}.
\end{aligned}$$

Umgeformt erhält man

$$\begin{aligned}
\cos(\varphi)\left(\frac{\partial g}{\partial r} - \frac{1}{r}\frac{\partial h}{\partial \varphi}\right) &= \sin(\varphi)\left(\frac{\partial h}{\partial r} + \frac{1}{r}\frac{\partial g}{\partial \varphi}\right), \\
\sin(\varphi)\left(\frac{\partial g}{\partial r} - \frac{1}{r}\frac{\partial h}{\partial \varphi}\right) &= -\cos(\varphi)\left(\frac{\partial h}{\partial r} + \frac{1}{r}\frac{\partial g}{\partial \varphi}\right).
\end{aligned}$$

Durch Quadrieren der beiden Gleichungen und Addieren erhält man

$$\left|\frac{\partial g}{\partial r} - \frac{1}{r}\frac{\partial h}{\partial \varphi}\right| = \left|\frac{\partial h}{\partial r} + \frac{1}{r}\frac{\partial g}{\partial \varphi}\right|.$$

Die beiden Gleichungen können also nur erfüllt werden, wenn die Ausdrücke in den Klammern 0 sind.

$$\Rightarrow \frac{\partial g}{\partial r} - \frac{1}{r}\frac{\partial h}{\partial \varphi} = 0 = \frac{\partial h}{\partial r} + \frac{1}{r}\frac{\partial g}{\partial \varphi}.$$

3. Aufgabe

1. $f(x+iy) = u(x,y) \in \mathbb{R}\ \forall x,y \in \mathbb{R}$. Dann sind die Cauchy-Riemann-Differentialgleichungen

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \quad \text{und} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0.$$

Also ist $f' \equiv 0$. Für beliebige $z \in \mathbb{C}$ folgt aus Satz 9

$$f(z) - f(0) = \int\limits_{L(0,z)} f'(z)\,\mathrm{d}z = 0$$

für die Strecke $L(0,z)$ von 0 nach z. Also ist $f(z) = f(0)\ \forall z \in \mathbb{C}$ und damit ist f konstant.

2. $f = u + iv$ holomorph ist äquivalent zu

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{und} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$\overline{f}$ holomorph ist äquivalent zu

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \quad \text{und} \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

Wenn f und $\overline{f}$ holomorph sein sollen, muss also gelten

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}$$

Also ist $f' \equiv 0$ und damit f konstant (siehe 1.).

4. Aufgabe

Es muss gelten:

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x + 2ay = \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= 2ax + 2by = -\frac{\partial v}{\partial x} \end{aligned}$$

Mit Integration folgt

$$\begin{aligned} v(x,y) &= 2xy + ay^2 + c(x), \\ v(x,y) &= -ax^2 - 2bxy + d(y), \end{aligned}$$

also gilt

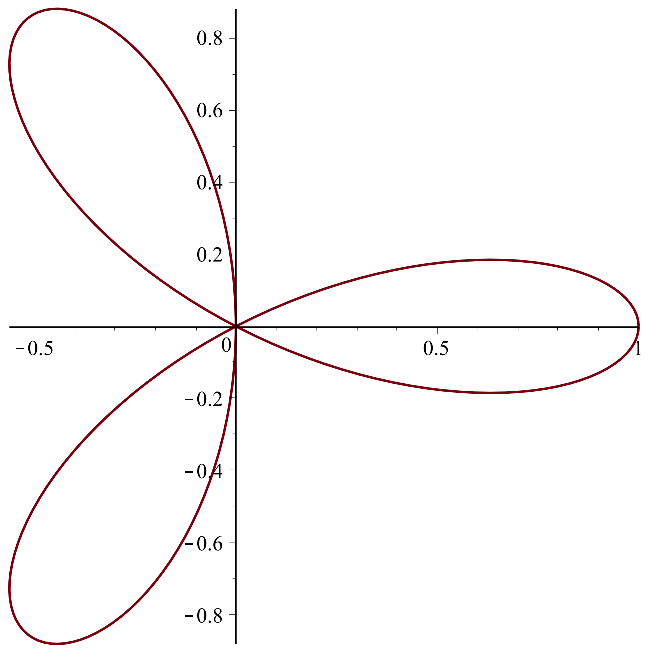
$$\begin{aligned} & 2xy + ay^2 + c(x) = -ax^2 - 2bxy + d(y) \\ \Leftrightarrow \quad & c(x) + ax^2 = d(y) - ay^2 - (2+2b)xy. \end{aligned}$$

Also ist $c(x) = -ax^2$, $d(y) = ay^2$ und $(2+2b)xy = 0$. Also muss $b = -1$ sein. Für $b = -1$, a beliebig und $v(x,y) = -ax^2 + 2xy + ay^2 + c$ ist f damit holomorph.
Alternative zur Bestimmung von b (aus Aufgabe 1, Teil 2):

$$\begin{aligned} & \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ \Leftrightarrow \quad & 2 + 2b = 0 \\ \Leftrightarrow \quad & b = -1. \end{aligned}$$

5. Aufgabe

1. $\cos(3t)e^{it} = \cos(3t)\cos(t) + i\cos(3t)\sin(t)$.



2. 1. Kreisbogen: $z_0 = \frac{1}{2},\ R = \frac{1}{2},\ \varphi_0 = \pi,\ \varphi_1 = 0$ (im Uhrzeigersinn)
   2. Kreisbogen: $z_0 = 0,\ R = 1,\ \varphi_0 = 0,\ \varphi_1 = \pi$ (gegen den Uhrzeigersinn)
   Strecke: $z_0 = -1,\ z_1 = 0$.
   Gesamte Kurve:

$$\gamma : [0,3] \to \mathbb{C} \quad , \quad \gamma(t) = \begin{cases} \frac{1}{2} + \frac{1}{2} e^{i\pi(1-t)} & , \quad t \in [0,1) \\ e^{i\pi(t-1)} & , \quad t \in [1,2) \\ t-3 & , \quad t \in [2,3] \end{cases} .$$

3.

$$\int_\gamma \frac{1}{z - \frac{1}{2}} \,\mathrm{d}z = \int_0^1 \frac{\dot\gamma(t)}{\gamma(t) - \frac{1}{2}} \,\mathrm{d}t + \int_1^2 \frac{\dot\gamma(t)}{\gamma(t) - \frac{1}{2}} \,\mathrm{d}t + \int_2^3 \frac{\dot\gamma(t)}{\gamma(t) - \frac{1}{2}} \,\mathrm{d}t.$$

$$\int_0^1 \frac{\dot\gamma(t)}{\gamma(t) - \frac{1}{2}} \,\mathrm{d}t = -i\pi \int_0^1 \frac{\frac{1}{2} e^{i\pi(1-t)}}{\frac{1}{2} e^{i\pi(1-t)}} \,\mathrm{d}t = -i\pi,$$

$$\begin{aligned} \int_1^2 \frac{\dot\gamma(t)}{\gamma(t) - \frac{1}{2}} \,\mathrm{d}t &= i\pi \int_1^2 \frac{e^{i\pi(t-1)}}{e^{i\pi(t-1)} - \frac{1}{2}} \,\mathrm{d}t = i\pi \int_0^1 \frac{e^{i\pi t}}{e^{i\pi t} - \frac{1}{2}} \,\mathrm{d}t = i\pi \int_0^1 \frac{e^{i\pi t}\left(e^{-i\pi t} - \frac{1}{2}\right)}{\left(e^{i\pi t} - \frac{1}{2}\right)\left(e^{-i\pi t} - \frac{1}{2}\right)} \,\mathrm{d}t \\ &= i\pi \int_0^1 \frac{1 - \frac{1}{2} e^{i\pi t}}{1 - \frac{1}{2} e^{i\pi t} - \frac{1}{2} e^{-i\pi t} + \frac{1}{4}} \,\mathrm{d}t = i\pi \int_0^1 \frac{1 - \frac{1}{2}\cos(\pi t)}{\frac{5}{4} - \cos(\pi t)} \,\mathrm{d}t + \pi \int_0^1 \frac{\frac{1}{2}\sin(\pi t)}{\frac{5}{4} - \cos(\pi t)} \,\mathrm{d}t \\ &= i\pi + \ln(3), \end{aligned}$$

$$\int_2^3 \frac{\dot\gamma(t)}{\gamma(t) - \frac{1}{2}} \,\mathrm{d}t = \int_2^3 \frac{1}{t - 3 - \frac{1}{2}} \,\mathrm{d}t = \int_2^3 \frac{1}{t - \frac{7}{2}} \,\mathrm{d}t = \left[\ln\left(\frac{7}{2} - t\right)\right]_2^3 = \ln\left(\frac{1}{2}\right) - \ln\left(\frac{3}{2}\right) = -\ln(3).$$

$$\int_\gamma \frac{1}{z - \frac{1}{2}} \,\mathrm{d}z = -i\pi + i\pi + \ln(3) - \ln(3) = 0.$$