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Solving linear operator equations in Banach spaces non-iteratively by the method of approximate inverse

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Abstract
The method of approximate inverse is a mollification method for stably solving inverse problems. In its original form it has been developed to solve operator equations in $L^2$-spaces and general Hilbert spaces. We show that the method of approximate inverse can be extended to solve linear, ill-posed problems in Banach spaces. This paper is restricted to function spaces. The method itself consists of evaluations of dual pairings of the given data with reconstruction kernels that are associated with mollifiers and the dual of the operator. We first define what we mean by a mollifier in general Banach spaces and then investigate two settings more exactly: the case of $L^p$-spaces and the case of the Banach space of continuous functions on a compact set. For both settings we present the criteria turning the method of approximate inverse into a regularization method and prove convergence with rates. As an application we refer to x-ray diffractometry which is a technique of non-destructive testing that is concerned with computing the stress tensor of a specimen. Since one knows that the stress tensor is smooth, x-ray diffractometry can appropriately be modelled by a Banach space setting using continuous functions.

1. Introduction

Solving linear operator equations

$$Af = g$$

for given measurement data $g$ is one of the fundamental tasks of applied mathematics, since the connection of a searched quantity $f$ and a measured quantity $g$ often is given, maybe after a linearization, by a linear, bounded operator. Such problems arise in industry, medical imaging and natural sciences. Often the mapping $A$ is compact and hence its inverse is unbounded and
a regularization is needed. By now in most articles that are dealing with the regularization of linear operators, a Hilbert space setting is used which allows us to apply all the useful tools like inner products, orthogonal complements and projections or generalized inverses. But for some applications, like e.g. the x-ray diffractometry that demands for smooth solutions, a Hilbert space setting like the usage of $L^2$-spaces, is not convenient. This is why in the last decade the extension of existing and well-established regularization methods in Hilbert spaces to Banach spaces is a lively field of research.

We want to name a few articles which are concerned with regularization methods in Banach spaces. Alber [2] was one of the first to investigate iterative regularization methods for Banach spaces. Butnariu et al [4] proved total convexity of norms in uniformly convex Banach spaces and that was an important starting point for the article [5] of Butnariu et al where the convergence of Bregman projection algorithms in Banach spaces was proven. Total variation-based image restoration was solved by an iterative regularization method by Osher et al [13]. The concept of Bregman distances and projections was used to extend the Landweber method to Banach spaces by Schöpfer et al [16], to construct sequential subspace methods [18] and split feasibility problems [17] and to investigate the minimization of Tikhonov functionals in Banach spaces, see Bonesky et al [3]. Once some iterative methods for inverse problems in Banach spaces were developed, there was an interest in investigating the convergence and finding rates of convergence. Hofmann et al [7] proved convergence rate results for non-smooth operators. Resmerita [15] obtained general convergence rates for regularization methods in Banach spaces. Düvelmeyer et al [6] investigated convergence results when using inexact source conditions. Kaltenbacher et al [8] considered the convergence of the Landweber method and the iteratively regularized Gauß–Newton method.

All the articles listed above have in common that they are dealing with iterative regularization methods. In this paper we define a non-iterative regularization method in Banach spaces by extending the well-known method of approximate inverse, established by Louis in [9, 10] to Banach spaces. The approximate inverse consists of the evaluation of the given and may be noise-contaminated data with precomputed reconstruction kernels. These reconstruction kernels are solutions of a dual equation that is associated with a so-called mollifier that implies the regularization property. The method is well established in Hilbert space settings and was successfully applied to different problems such as computerized tomography, Louis, Schuster [12], inverse scattering, Abdullah, Louis [1], Doppler tomography [19, 20, 24], x-ray diffractometry [23] and even image processing, Louis [11]. A concise monograph on that method is Schuster [22]. In this paper we extend the method to Banach spaces, show conditions to get convergence rates $O(\delta^{1-\nu})$ where $\delta$ represents the noise level and state these considerations for $L^p$-spaces and the space $C(K)$ of continuous functions on a compact set $K$. Finally we outline the concepts on x-ray diffractometry which is a specific method of non-destructive testing.

The paper is organized as follows. We at first define what we mean by a regularization method for a linear operator equation $Af = g$ on a Banach space and then introduce to the method of approximate inverse and show how operator invariances of $A$ can be helpful to reduce the computational effort for the calculation of the reconstruction kernels. In all these considerations we assume that $A$ is one-to-one. We derive the result that $O(\delta)$ is the optimal convergence rate in an asymptotic meaning. For $L^p$-settings, where $1 \leq p < \infty$, and the $C(K)$-setting we are able to prove conditions to get the rate $O(\delta^{1-\nu})$ for arbitrary $0 < \nu < 1$. It might be remarkable that we do not use source conditions such as that the exact solution $f$ is in the range of some operator associated with $A$ as $f \in R([A^*A]^{1/2})$ in a Hilbert space setting, but rather that $f$ has to be sufficiently smooth and the mollifier has additionally a sufficiently
large order. We at last show convergence rates for the application of approximate inverse in x-ray diffractometry.

2. Mollifiers and reconstruction kernels in Banach spaces

Let \( X, Y \) be Banach spaces with norms \( \| \cdot \|_X, \| \cdot \|_Y \) and \( A : X \to Y \) a linear, bounded and injective operator. With \( X^*, Y^* \) we denote the duals of \( X \) and \( Y \), respectively, and

\[
\langle f^*, f \rangle_{X^* \times X} = f^*(f), \quad \langle g^*, g \rangle_{Y^* \times Y} = g^*(g)
\]
denote dual pairings on \( X^* \times X \) and \( Y^* \times Y \), respectively. Equipped with the norms

\[
\| f^* \|_{X^*} = \operatorname{sup}_{f \in B_X} | f^*(f) |, \quad \| g^* \|_{Y^*} = \operatorname{sup}_{g \in B_Y} | g^*(g) |
\]
where \( B_X \) and \( B_Y \) are the closed unit balls in \( X \) and \( Y \), respectively, the duals \( X^* \) and \( Y^* \) turn themselves into Banach spaces. The adjoint mapping \( A^* \) is then a linear and bounded operator between the Banach spaces \( Y^* \) and \( X^* \) which is characterized by

\[
\langle A^* g^*, f \rangle_{X^* \times X} = \langle g^*, A f \rangle_{Y^* \times Y}.
\]

We aim for solving the equation

\[
Af = g^\delta \quad (1)
\]
with given noise-contaminated data \( g^\delta \) satisfying

\[
\| g - g^\delta \|_Y < \delta.
\]
If the range of \( A \) is not closed, then the inverse \( A^{-1} \) is not continuous and hence the solution of (1) is unstable. In this case we need a regularization method which is a family \( \{ R_\gamma \} \) of bounded operators between \( Y \) and \( X \) that converge pointwise to \( A^{-1} \). We at first precise the concept of regularization.

**Definition 1.** A family \( \{ R_\gamma \} \) of linear and bounded operators \( R_\gamma : Y \to X \) is called a regularization method for a mapping \( A \in L(X, Y) \), if for given \( g \in \mathcal{R}(A) \) there is a parameter choice rule \( \gamma = \gamma(\delta, g^\delta) \) such that

\[
\lim_{\delta \to 0} \operatorname{sup}_{\gamma(\delta, g^\delta)} [ \| g - g^\delta \|_Y < \delta ] = 0
\]
satisfying

\[
\lim_{\delta \to 0} \operatorname{sup}_{\gamma(\delta, g^\delta)} [ \| A^{-1} g - R_{\gamma(\delta, g^\delta)} g^\delta \|_X : \| g - g^\delta \|_Y < \delta ] = 0.
\]

In this paper we only consider the Banach spaces \( X, Y \) consisting of functions on a domain \( \Omega \subset \mathbb{R}^d \). As typical examples we refer to \( L^p \)-spaces with \( 1 \leq p \leq \infty \) or the Banach space of functions which are continuous on a compact set. There exist regularization methods on Banach spaces as the minimization of Tikhonov–Phillips functionals, the Landweber method, Gauß–Newton or iteratively regularized Gauß–Newton methods amongst others. In this paper we construct a regularization method for linear operators in Banach spaces with the help of so-called mollifiers which we introduce next.

**Definition 2.** Let \( X \) be a Banach space consisting of functions with the domain \( \Omega \subset \mathbb{R}^d \). We call a family \( \{ e_\gamma \} \) of mappings \( e_\gamma : \Omega \to X^* \) a mollifier for \( X \) (or \( X \)-mollifier), if the following two conditions hold true.
(a) For each function \( f \in X \), the family \( \{ f_{\gamma} \} \) of mappings
\[
f_{\gamma}(x) := \langle e_{\gamma}(x), f \rangle_{X^* \times X}
\]
belongs to \( X \).
(b) The functions \( f_{\gamma} \) converge to \( f \) in \( X \), that is
\[
\lim_{\gamma \to 0} \| f_{\gamma} - f \|_X = 0, \quad f \in X.
\]

We will show specific criteria for such a family \( \{ e_{\gamma} \} \) to be \( L^p(\Omega) \)- and \( C(K) \)-mollifiers. Once we have chosen a mollifier \( \{ e_{\gamma} \} \), we associate a reconstruction kernel with it. A reconstruction kernel is a family \( \{ v_{\gamma} \} \) of mappings \( v_{\gamma} : \Omega \to Y^* \) solving the dual equation
\[
A^* v_{\gamma}(x) = e_{\gamma}(x), \quad x \in \Omega.
\]

In the following we assume that \( e_{\gamma}(x) \) is in the range of \( A^* \). If we have the reconstruction kernel \( v_{\gamma} \) at hand, then we are able to formulate the method of approximate inverse in Banach spaces.

**Definition 3.** Let \( A \in L(X, Y) \), \( \{ e_{\gamma} : \Omega \to X^* \} \) be a mollifier and \( \{ v_{\gamma} : \Omega \to Y^* \} \) the corresponding reconstruction kernel according to (4). The family of mappings \( \{ A_{\gamma} \} \) defined by
\[
A_{\gamma} g(x) := \langle v_{\gamma}(x), g \rangle_{Y^* \times Y}, \quad g \in Y, \quad x \in \Omega,
\]
is called the approximate inverse of \( A \) associated with the mollifier \( \{ e_{\gamma} \} \).

We shortly say that \( A_{\gamma} \) is the approximate inverse of \( A \) if it is clear which mollifier we have chosen. Obviously \( A_{\gamma} \) is a linear mapping from \( Y \) into the vector space of real-valued functions on \( \Omega \) and weak convergence of a sequence \( \{ g_n \} \subset Y \) implies pointwise convergence of \( \{ A_{\gamma} g_n \} \). Moreover if \( A f = g \), then from (4) we may deduce that
\[
A_{\gamma} g(x) = \langle v_{\gamma}(x), g \rangle_{Y^* \times Y} = \langle e_{\gamma}(x), f \rangle_{X^* \times X}
\]
and (3) shows that
\[
\lim_{\gamma \to 0} A_{\gamma} g = f
\]
with respect to the strong (norm-) topology in \( X \). In that sense \( A_{\gamma} \) in fact is an approximate inverse to \( A \). Under specific assumptions it turns out to be a regularization method.

**Theorem 1.** Let the assumptions of definition 3 hold true. Assume that \( A \) is injective, that the mappings \( A_{\gamma} \) map \( Y \) into \( X \) and are bounded with estimates
\[
\| A_{\gamma} \|_{Y \to X} \leq l_{\gamma}.
\]
Let \( f \in X \) be the solution of \( A f = g \) and \( g^\delta \in Y \) be noise-contaminated data with
\[
\| g^\delta - g \|_Y < \delta.
\]
If the parameter choice rule \( \gamma = \gamma(\delta) \) is chosen such that \( \gamma(\delta) \to 0 \) for \( \delta \to 0 \) and
\[
l_{\gamma(\delta)} = o(\delta^{-1}),
\]
then
\[
\lim_{\delta \to 0} \| f_{\gamma(\delta)}^\delta - f \|_X = 0,
\]
where \( f_{\gamma(\delta)}^\delta = A_{\gamma(\delta)} g^\delta \). That means the approximate inverse \( \{ A_{\gamma} \} \) represents a regularization method.
Proof. To prove the regularization property we estimate
\[ \| f_\gamma - f \|_X \leq \| A_\gamma g - f \|_X + \| A_\gamma (g^\delta - g) \|_X \]
\[ \leq \| A_\gamma g - f \|_X + l_\gamma \delta. \]
The first summand tends to 0 as \( \gamma \to 0 \) because of (5); the second summand tends to zero because of (7).

\[ \square \]

From the proof of theorem 1 we can easily deduce convergence rates.

Corollary 1. Let the assumptions of theorem 1 hold and \( f_\gamma = A_\gamma g \) be the approximate inverse applied to exact data. If the parameter choice rule \( \gamma = \gamma(\delta) \) is chosen according to
\[ \lim_{\delta \to 0} \gamma(\delta) = 0, \quad l_\gamma(\delta) = O(\delta^{-v}), \quad 0 < v < 1 \]
then
\[ \| f_\gamma(\delta) - f \|_X = O(\delta^{1-v}) \quad \text{as} \quad \delta \to 0. \]

As in Hilbert spaces, see [9], operator invariances can be useful to enhance the efficacy of the computation of the reconstruction kernels.

Lemma 1. If to each \( x \in \Omega \) there exist mappings \( T_1^x \in L(X^*) \), \( T_2^x \in L(Y^*) \) and \( x_0 \in \Omega \) such that
\[ e_\gamma(x) = T_1^x e_\gamma(x_0) \quad \text{and} \quad T_1^x A^* = A^* T_2^x, \quad x \in \Omega, \quad (8) \]
then
\[ v_\gamma(x) = T_2^x v_\gamma(x_0). \quad (9) \]

Proof. Assertion (9) becomes obvious from (4) and (8).

Hence it suffices to compute one single reconstruction kernel \( v_\gamma(x_0) \) to generate all reconstruction kernels applying \( T_2^x \) to it, if only (8) is satisfied.

3. The case \( X = L^p(\Omega) \)

We consider the specific situation where \( X = L^p(\Omega) \) with \( 1 \leq p < \infty \) and \( \Omega \subset \mathbb{R}^d \) is an open subset. For \( p \) we denote by \( p' \) the dual exponent, which means \( 1 \leq p' \leq \infty \) with
\[ 1 + \frac{1}{p} = 1. \]
For \( p = 1 \) we set \( p' = \infty \). Then \( X^* = L^{p'}(\Omega) \) and the dual pairing is given by
\[ (f^*, f)_{L^{p'} \times L^p} = \int_\Omega f^*(x) f(x) \, dx. \]
We want to present a recipe to construct mollifiers for that situation. Furthermore we prove that under certain conditions
\[ \| f_\gamma - f \|_{L^p(\Omega)} \leq C_\gamma \| f \|_{C^0(\Omega)} \quad \text{for all} \quad \gamma > 0, \]
where \( \Omega \) is bounded. This estimate is sufficient to achieve the convergence rate \( O(\delta^{1-v}) \), see corollary 1.

We present now a recipe to generate mollifiers in bounded domains \( \Omega \).
Theorem 2. Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain, \( \tilde{e} \in L^p(\mathbb{R}^d) \) be a function with
\[
\text{ess sup} (\tilde{e}) = B_{\rho}(0) \subset \Omega_1
\]
and
\[
\int_{\Omega} \tilde{e}(x) \, dx = \int_{B_{\rho}(0)} \tilde{e}(x) \, dx = 1. \tag{10}
\]
Define
\[
e_{\gamma}(x, y) := \gamma^{-d} \tilde{e}\left(\frac{x - y}{\gamma}\right) \quad a.e. \tag{11}
\]
Then, \( \{e_{\gamma}\} \) is a mollifier for \( L^p(\Omega) \).

Proof. Since \( \Omega \) is bounded, we may estimate for \( k_{\gamma}(x) \):
\[
\|k_{\gamma}\|_{L^p}^p = \int_{\Omega} \|k_{\gamma}(x)\|^p \, dx = \int_{\Omega} \|e_{\gamma}(x, \cdot)\|^p_{L^p} \, dx
\]
\[
\leq \int_{\Omega} \left( \int_{\mathbb{R}^d} \left| \gamma^{-d} \tilde{e}\left(\frac{x - y}{\gamma}\right) \right|^{\frac{p}{p'}} \, dy \right)^{p/p'} \, dx
\]
\[
= \gamma^{-d} \int_{\Omega} \left( \int_{\mathbb{R}^d} |\tilde{e}(z)|^{p'} \, dz \right)^{p/p'} \, dx = \gamma^{-d} \|\tilde{e}\|_{L^p(\mathbb{R}^d)}^{p/p'}, \tag{12}
\]
where we applied the substitution \( z \leftarrow (x - y)/\gamma \). Hence, \( k_{\gamma} \in L^p \) and this implies \( f_{\gamma} \in L^p \).

It remains to show the convergence \( f_{\gamma} \to f \) in \( L^p \). To this end we first assume that \( f \in C_0(\Omega) \) is a continuous function with compact support in \( \Omega \) and then use a density argument. Hence, let \( f \in C_0(\Omega) \). We have
\[
\|f_{\gamma} - f\|_{L^p}^p = \int_{\Omega} \left| f(x) - \int_{\Omega} \gamma^{-d} f(y) \tilde{e}\left(\frac{x - y}{\gamma}\right) \, dy \right|^p \, dx.
\]
We first investigate the integrand and estimate
\[
I_{\gamma}(x) = \left| f(x) - \int_{\Omega} \gamma^{-d} f(y) \tilde{e}\left(\frac{x - y}{\gamma}\right) \, dy \right|
\]
\[
= \left| f(x) - \int_{B_{\rho}(0)} \gamma^{-d} f(y) \tilde{e}\left(\frac{x - y}{\gamma}\right) \, dy \right|
\]
\[
= \left| f(x) - \int_{B_{\rho}(0)} f(x - \gamma z) \tilde{e}(z) \, dz \right|
\]
\[
= \int_{B_{\rho}(0)} \left| f(x) - f(x - \gamma z) \right| \tilde{e}(z) \, dz \leq \left( \int_{B_{\rho}(0)} \left| f(x) - f(x - \gamma z) \right|^p \, dz \right)^{1/p} \left( \int_{B_{\rho}(0)} |\tilde{e}(z)|^p \, dz \right)^{1/p'}
\]
\[
= \|\tilde{e}\|_{L^p} \left( \int_{B_{\rho}(0)} \left| f(x) - f(x - \gamma z) \right|^p \, dz \right)^{1/p}
\]
using Hölder’s inequality and the substitution \( z \leftarrow (x - y)/\gamma \). Since \( |a + b|^p \leq 2^p (|a|^p + |b|^p) \) for the real values \( a, b \) we have that
\[
\left| f(x) - f(x - \gamma z) \right|^p \leq 2^p (|f(x)|^p + |f(x - \gamma z)|^p).
\]
Furthermore the function
\[
\tilde{f}(x) := \sup_{z \in B_{\rho}(0)} |f(x - \gamma z)|
\]
is a continuous function with compact support satisfying
\[ f(x) = |f(x - y)|. \]

As a summary of our results we obtain

(i) \( I_\gamma(x)^p \to 0 \) pointwise as \( \gamma \to 0 \),

(ii) \( I_\gamma(x)^p \leq \|e_\gamma(x)\|_{L^p}^p |B_\rho(0)|2^p(|f(x)|^p + \tilde{f}(x)^p) \) and the right-hand side of that estimation is in \( L^1(\Omega) \).

Lebesgue’s theorem of dominated convergence finally implies
\[
\lim_{\gamma \to 0} \int_{\Omega} I_\gamma(x)^p \, dx = 0.
\]

Let now \( f \in L^p(\Omega) \) be arbitrary and \( k_\gamma := \|e_\gamma\|_{L^p}^p \). Since \( C_0(\Omega) \) is dense in \( L^p(\Omega) \) we find a sequence \( \{f_n\} \subset C_0(\Omega) \) such that \( f_n \to f \) in \( L^p \) as \( n \to \infty \). Moreover we choose a sequence \( \{n(\gamma')\} \subset \mathbb{N} \) such that \( n(\gamma') \to \infty \) as \( \gamma' \to 0 \) and
\[
k_{\gamma}^{-1} \gamma^T \to 0 \quad \text{as} \quad \gamma \to 0,
\]
where \( \tau > 0 \) is sufficiently large. Such \( \tau \) exists since \( k_{\gamma}^{-1} \gamma^T \) grows polynomially in \( \gamma \), see (12).

Let \( \varepsilon > 0 \) be arbitrary and \( \gamma_0 > 0 \) be such that
\[
k_{\gamma}^{-1} \gamma^T < \frac{\varepsilon}{5} \quad \text{and} \quad \gamma' < \frac{\varepsilon}{5} \quad \text{for all} \quad \gamma < \gamma_0
\]
(such \( \gamma_0 \) exists because of (13)) and
\[
\|f_{n(\gamma')} - f_{n(\gamma_0)}\|_{L^P} < k_{\gamma}^{-1} \gamma^T \quad \text{for all} \quad \gamma < \gamma_0
\]
(this can be fulfilled since \( \{f_n\} \) is convergent and hence a Cauchy sequence). Furthermore let \( 0 < \gamma_1 < \gamma_0 \) be such that
\[
\|e_\gamma(x, \cdot), f_{n(\gamma_0)}(x)\|_{L^p} < \frac{\varepsilon}{5} \quad \text{for all} \quad \gamma < \gamma_1
\]
(such \( \gamma_1 \) exists, since \( f_{n(\gamma_0)} \) is in \( C_0(\Omega) \)). We estimate
\[
\|f_\gamma - f\|_{L^p} \leq \|e_\gamma(x, \cdot), f - f_{n(\gamma')}\|_{L^p} + \|e_\gamma(x, \cdot), f_{n(\gamma')} - f_{n(\gamma_0)}(x)\|_{L^p} + \|f_{n(\gamma_0)} - f_{n(\gamma)}\|_{L^p} + \|f_{n(\gamma)} - f\|_{L^p}.
\]

Further estimation shows
\[
\|e_\gamma(x, \cdot), f - f_{n(\gamma')}\|_{L^p} \leq \|k_\gamma\|_{L^p} \|f - f_{n(\gamma')}\|_{L^p} \leq \gamma^T,
\]
\[
\|e_\gamma(x, \cdot), f_{n(\gamma')} - f_{n(\gamma_0)}(x)\|_{L^p} \leq \|k_\gamma\|_{L^p} \|f_{n(\gamma)} - f_{n(\gamma_0)}\|_{L^p} \leq \gamma^T
\]
because of (13). Hence, for \( \gamma < \gamma_1 \) each of the summands is less than \( \varepsilon/5 \) and thus
\[
\|f_\gamma - f\|_{L^p} < \varepsilon \quad \text{as} \quad \gamma < \gamma_1,
\]
which finishes the proof. \( \square \)

Another criterion for \( f_\gamma \) to be an \( L^p \)-function is the boundedness of \( k_\gamma \).

**Lemma 2.** Let \( \{e_\gamma : \Omega \to L^p(\Omega)\} \) be a family of mappings such that the function
\[
k_\gamma(x) := \|e_\gamma(x)\|_{L^p} \in L^p.
\]
Then,
\[
f_\gamma(x) = \langle e_\gamma(x), f \rangle_{L^p} \in L^p.
\]
Condition (14) is satisfied if \( k_\gamma \in L^\infty(\Omega) \), where in the case when \( \Omega \) is unbounded we have to postulate additionally
\[
k_\gamma(x) \leq c|x|^s \quad \text{for} \quad x \in \{x \in \mathbb{R}^d : |x| > R \} \cap \Omega
\]
for the constants \( c, R > 0 \) and \( s < -d/p \).

**Proof.** Let \( f \in L^p \). The estimate
\[
\int_{\Omega} |f_\gamma(x)|^p \, dx = \int_{\Omega} |(e_\gamma(x), f)_{L^p \times L^p}|^p \, dx \\
\leq \|f\|_{L^p}^p \int_{\Omega} \|e_\gamma(x)\|_{L^p}^p \, dx = \|f\|_{L^p}^p \|k_\gamma\|_{L^p}^p
\]
and (14) implies \( f_\gamma \in L^p \).

If \( \Omega \) is bounded and \( k_\gamma \in L^\infty(\Omega) \), then obviously (14) holds true. Let now \( \Omega \) be unbounded. We have
\[
\int_{\Omega} |k_\gamma(x)|^p \, dx = \int_{\Omega \cap B_\varepsilon(0)} |k_\gamma(x)|^p \, dx + \int_{\Omega \cap \{x \in \mathbb{R}^d : |x| > R\}} |k_\gamma(x)|^p \, dx \\
\leq \int_{\Omega \cap B_\varepsilon(0)} |k_\gamma(x)|^p \, dx + c^p \int_{\Omega \cap \{x \in \mathbb{R}^d : |x| > R\}} |x|^sp \, dx.
\]
The first integral is bounded since \( k_\gamma \in L^\infty(\Omega) \). The second integral can be estimated by using the spherical coordinates \( x = r\omega, r > 0, \omega \in S^{d-1} \) and
\[
\int_{\Omega \cap \{x \in \mathbb{R}^d : |x| > R\}} |x|^sp \, dx \leq \int_{\{x \in \mathbb{R}^d : |x| > R\}} |x|^sp \, dx \\
= |S^{d-1}| \int_R^\infty r^{d-1} r^{sp} \, dr < \infty,
\]
since \( sp + d - 1 < -1 \iff s < -d/p \). Here, \(|S^{d-1}|\) denotes the surface measure of the unit sphere in \( \mathbb{R}^d \).

We finish this section by proving rates for the convergence \( f_\gamma \to f \) in the case when \( \Omega \) is bounded and the mollifier \( \{e_\gamma\} \) is generated by (11). These rates are important to get the stability rates in corollary 1. To obtain these convergence rates a momentum condition on \( \tilde{e} \) is required.

**Definition 4.** Let \( \tilde{e} \in L^p(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \). We say that \( \tilde{e} \) has order \( m \) if the following three conditions hold true:

(a) \[
\int_{\mathbb{R}^d} \tilde{e}(z) \, dz = 1,
\]

(b) \[
\int_{\mathbb{R}^d} \tilde{e}(z) z^\alpha \, dz = 0
\]
for all multi-indices \( \alpha \in \mathbb{N}_0^d \) with \( 1 \leq |\alpha| < m \),

(c) \[
\mu_\alpha := \int_{\mathbb{R}^d} \tilde{e}(z) z^\alpha \, dz / \alpha! \neq 0
\]
for all multi-indices \( \alpha \in \mathbb{N}_0^d \) with \( |\alpha| = m \).
Now we have all ingredients together to prove convergence rates for $f_\gamma$ if we apply mollifiers that are constructed due to theorem 2.

**Theorem 3.** Adopt the assumptions of theorem 2, where additionally $\tilde{e}$ has order $m \in \mathbb{N}$ and $\text{ess sup} (\tilde{e}) \subseteq B_\rho(0) \subset \Omega$. Let further $f \in C^m_0(\Omega)$. Then there exists a constant $C > 0$ such that

$$
\| f_\gamma - f \|_{L^p} \leq C \gamma^m \| f \|_{C^m(\Omega)} \quad \text{for all } \gamma > 0.
$$

**Proof.** We show (17) for $d = 1$ only, since this assertion does not depend on the dimension. Then $B_\rho(0) = (-\rho, \rho)$ and we have

$$
\| f_\gamma - f \|_{L^p} = \int_{\Omega} I_\gamma(x)^p \, dx,
$$

with $I_\gamma(x)$ as in the proof of theorem 2. Using the momentum conditions for $\tilde{e}$ and the Taylor approximation

$$
f(x - \gamma z) = \sum_{j=0}^{m-1} \frac{f^{(j)}(x)}{j!} (-\gamma z)^j + \frac{f^{(m)}(x_{\gamma,z})}{m!} (-\gamma z)^m,
$$

where $x_{\gamma,z}$ denotes an intermediate value between $x$ and $x - \gamma z$, we compute

$$
I_\gamma(x) = \left| \int_{-\rho}^{\rho} [f(x) - f(x - \gamma z)] \tilde{e}(z) \, dz \right| = \left| \int_{-\rho}^{\rho} (-\gamma z)^m \frac{m!}{m!} f^{(m)}(x_{\gamma,z}) \tilde{e}(z) \, dz \right| \leq \gamma^m K_m \| f^{(m)} \|_{\infty},
$$

where

$$
K_m := \int_{-\rho}^{\rho} |\tilde{e}(z) z^m| \, dz.
$$

From this we get

$$
\| f_\gamma - f \|_{L^p} = \int_{\Omega} I_\gamma(x)^p \, dx \leq \gamma^m K_m \| f^{(m)} \|_{\infty} \| f \|_{C^m(\Omega)}
$$

for all $\gamma > 0$. Hence, (17) is satisfied with $C := |\Omega|^{1/p} K_m$. \hfill \Box

**Remark 1.** By now there is no proof that (17) is valid as $\gamma \to 0$ for $f \in W^{p,m}(\Omega)$ since the constant $C$ involves the sup-norm of $f$. Thus, a density argument does not help.

With the help of theorem 3 we are able to formulate conditions for $\tilde{e}$ and $f$ to obtain the convergence rate $O(\delta^{-1-\nu})$ of the approximate inverse $f_\gamma^d = A_\gamma g^d$ in $L^p$-spaces.

**Theorem 4.** Let the assumptions of theorems 1 and 3 hold true and $\nu$ be given with $0 < \nu < 1$. Furthermore assume that $a \asymp b$ means that there are constants $c_1, c_2 > 0$ such that $c_1 b \leq a \leq c_2 b$.

$$
I_\gamma \asymp \gamma^{-k} \quad \text{as } \gamma \to 0, \quad k > 0.
$$

If the parameter choice rule $\gamma = \gamma(\delta)$ is chosen such that

$$
\gamma(\delta) = O(\delta^{\nu/k}) \quad \text{as } \delta \to 0,
$$

(18)
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\[ f \in C^m_0(\Omega) \text{ and } \bar{e} \text{ has order } m, \text{ where } m \in \mathbb{N} \text{ satisfies} \]
\[ m \geq k \frac{1 - \nu}{\nu}, \quad (19) \]

then
\[ \| f^\delta_{\gamma(\delta)} - f \|_{L^p} \leq c \| f \|_{C^m(\Omega)} \quad \text{as } \delta \to 0. \]

**Proof.** Because of (18) we have that \( l_{\gamma(\delta)} = O(\delta^{-\nu}) \). From corollary 1 and (17) we deduce that
\[ \| f^\delta_{\gamma(\delta)} - f \|_{L^p} \leq \| f_{\gamma(\delta)} - f \|_{L^p} + l_{\gamma(\delta)} \delta \]
\[ \leq C (\delta^{-\nu} \| f \|_{C^m(\Omega)} + \delta^{1-\nu}). \]

Balancing the terms we arrive at
\[ \| f^\delta_{\gamma(\delta)} - f \|_{L^p} \leq c \delta^{-\nu} \| f \|_{C^m(\Omega)} \quad \text{as } \delta \to 0, \]

if \( m\nu/k \geq 1 - \nu \) which gives (19). \( \square \)

**Remark 2.** Condition (19) shows that the order of \( \bar{e} \) as well as the smoothness of the solution \( f \) has to be very large in order to approach in an asymptotic sense the optimal convergence rate \( O(\delta^\gamma) \), that is approached but never achieved as \( \nu \to 0 \). Hence a high convergence rate corresponds to strong assumptions on \( \bar{e} \) and \( f \). It is remarkable that in contrast to other regularization methods we have to postulate conditions not only to \( f \) but also to the mollifier \( \bar{e} \) which is independent from the range of \( A \).

4. The case \( X = C(K), K \text{ compact} \)

Let \( K \subset \mathbb{R}^d \) be a nonempty compact set with int \( (K) \neq \emptyset \) and \( C(K) \) be the Banach space of continuous functions defined on \( K \) equipped with the sup-norm
\[ \| f \|_\infty = \| f \|_{\infty,K} = \sup_{x \in K} | f(x) |. \]

We want to describe mollifiers and derive convergence rates in this situation. As an analogue to (17), we prove the estimate
\[ \| f^\gamma - f \|_\infty \leq C \gamma^m \| f \|_{C^m(K)}, \]

which again is sufficient to show convergence with rates in the case of noisy data \( g^\delta \).

The dual space \( C(K)^* \) of \( C(K) \) can be identified with the space of all regular and countable additive Borel measures on \( K \)
\[ \text{rca}(K) := \{ \lambda : \lambda \text{ is a regular and countable additive Borel measure on } B(K) \} \]
equipped with the total variation
\[ \| \lambda \|_{B} = \sup \left\{ \sum_{i=1}^{k} |\lambda(K_i)| : k \in \mathbb{N}, K_i \in B(K), K_i \text{ pairwise disjoint} \right\} \]
as norm, where \( B(K) \) denotes the \( \sigma \)-algebra of all Borel sets of \( K \), that is the \( \sigma \)-algebra which is generated by the closed subsets of \( K \). We have an isometric isomorphism \( C(K)^* \cong \text{rca}(K) \) and
\[ \lambda(E) = \int_E d\lambda(x) \quad \text{for all } E \subset B(K). \]
In that sense the dual pairing on $X$ is given by
\[ \langle \lambda, f \rangle_{\text{rca}(K) \times \mathcal{C}(K)} = \int_K f(x) \, d\lambda(x). \]

Note that any function $f^* \in L^1(K)$ generates a measure $\lambda_{f^*} \in \text{rca}(K)$ via
\[ \lambda_{f^*}(E) := \int_E f^*(x) \, dx, \quad E \in \mathcal{B}(K). \]
We present a concept to generate $\mathcal{C}(K)$-mollifiers which is similar to (11).

**Theorem 5.** Let $\lambda \in \text{rca}(K)$ be non-negative and suppose that
\[ \lambda(K) = \int_K d\lambda(x) = 1. \]  
(20)

We define a family of mappings $\{\lambda_\gamma : K \to \text{rca}(K)\}$ by
\[ \int_K f(x) \, d[\lambda_\gamma(y)](x) := \int_K f(\gamma x + y) \, d\tilde{\lambda}(x), \quad f \in \mathcal{C}(K). \]  
(21)

Then the family $\{\lambda_\gamma\}$ represents a mollifier for $\mathcal{C}(K)$.

Note that in (21) we silently set $f(z) := 0$ for all $z \in \mathbb{R}^d \setminus K$ which we assume in this section without always mentioning it. Definition (21) is then well defined since the translated function $f(\gamma x + y)$ is then $\tilde{\lambda}$-measurable for all $y \in K$.

**Proof.** At first we show that $\lambda_\gamma(y) \in \text{rca}(K)$ for each $y \in K$. Let $f \in \mathcal{C}(K)$. For $\gamma > 0$ and $y \in K$ fixed we have
\[ \sup_{x \in K} |f(\gamma x + y)| \leqslant \|f\|_{\infty}. \]

Defining $Tf := \int_K f(x) \, d[\lambda_\gamma(y)](x)$ we see that $T : \mathcal{C}(K) \to \mathbb{R}$ is linear. Since
\[ |Tf| = \left| \int_K f(x) \, d[\lambda_\gamma(y)](x) \right| = \left| \int_K f(\gamma x + y) \, d\tilde{\lambda}(x) \right| \leqslant \|\tilde{\lambda}\|_0 \|f\|_{\infty} \]
we have that $T$ is also bounded. Hence $T \in \mathcal{C}(K)^*$ and thus $\lambda_\gamma(y) \in \text{rca}(K)$.

Next we show that $f_\gamma(y) = (\lambda_\gamma(y), f)_{\text{rca}(K) \times \mathcal{C}(K)}$ is in $\mathcal{C}(K)$ for $f \in \mathcal{C}(K)$. Let $\varepsilon > 0$ be given. Since $K$ is compact, $f$ is even uniformly continuous. Hence there exists a $\eta = \eta(\varepsilon) > 0$ such that $|f(y) - f(y')| < \varepsilon$ whenever $\|y - y'\| < \eta$ for $y, y' \in K$. This implies
\[ |f_\gamma(y) - f_\gamma(y')| \leqslant \int_K |f(\gamma x + y) - f(\gamma x + y')| \, d\tilde{\lambda}(x) \]
\[ < \int_K \varepsilon \, d\tilde{\lambda}(x) = \varepsilon \quad y, y' \in K \]
proving the (uniform) continuity of $f_\gamma$, where in the latter estimate we used the fact that $\tilde{\lambda}$ is non-negative.

It remains to show the convergence $f_\gamma \to f$ in $\mathcal{C}(K)$ as $\gamma \to 0$. Again let $\varepsilon > 0$ and $\eta = \eta(\varepsilon)$ as above. We set $M := \sup_{x \in K} |x|$ and choose $\gamma_0 > 0$ such that $\gamma_0 M < \eta$. Such $\gamma_0$ exists since $M < \infty$ because of the compactness of $K$. Taking into account that $\tilde{\lambda}(K) = 1$, we may now estimate for $\gamma < \gamma_0$
\[ \|f_\gamma - f\|_{\infty} = \sup_{y \in K} \left| \int_K (f(\gamma x + y) - f(y)) \, d\tilde{\lambda}(x) \right| \]
\[ \leqslant \sup_{y \in K} \int_K |f(\gamma x + y) - f(y)| \, d\tilde{\lambda}(x) \]
\[ < \sup_{y \in K} \int_K \varepsilon \, d\tilde{\lambda}(x) = \varepsilon, \]
since $|\gamma x + y - y| = \gamma \|x\| \leq \gamma M < \eta$ for $\gamma < \gamma_0$. This proves that $f_{\gamma} \to f$ in $C(K)$ as $\gamma \to 0$. \hfill \Box

Remark 3. As mentioned before any function $f^* \in L^1(K)$ generates a measure $\tilde{\lambda}_{f^*} \in \text{rca}(K)$ via

$$\tilde{\lambda}_{f^*}(E) = \int_E f^*(x) \, dx, \quad E \in B(K).$$

In this case (20) means that $\int_K f^*(x) \, dx = 1$ and (21) reads

$$f^*_{\gamma}(x, y) = \gamma^{-d} f^*(\frac{x - y}{\gamma}),$$

which is the same construction as in theorem 2.

A further criterion for general measures $\{\lambda_{\gamma}\} \subset \text{rca}(K)$ leading to continuous functions $f_{\gamma}$ is the continuity of the mappings $y \mapsto \lambda_{\gamma}(y)$.

Lemma 3. If the mappings $\lambda_{\gamma} : K \to \text{rca}(K)$ are continuous as functions from $K$ to $\text{rca}(K)$, then $f_{\gamma} \in C(K)$ for each $f \in C(K)$. The functions $f_{\gamma}$ are then even uniformly continuous.

Proof. The assertions become clear from

$$|f_{\gamma}(x) - f_{\gamma}(y)| = |(\lambda_{\gamma}(x) - \lambda_{\gamma}(y), f_{\text{rca}(K) \times C(K)})| \leq \|\lambda_{\gamma}(x) - \lambda_{\gamma}(y)\|_{B(K)} \|f\|_{\infty}.$$ 

Because $K$ is compact, the functions $f_{\gamma}$ are even uniformly continuous. \hfill \Box

To show convergence with rates for $f_{\gamma} \to f$ we again have to define what we mean by the order of a measure $\lambda \in \text{rca}(K)$.

Definition 5. Let $\tilde{\lambda} \in \text{rca}(K)$, $K \subset \mathbb{R}^d$ compact and $m > 0$ an integer. We say that $\tilde{\lambda}$ has order $m$, if the following three conditions hold true:

(a)

$$\tilde{\lambda}(K) = 1,$$

(b)

$$\int_K x^\alpha \, d\tilde{\lambda}(x) = 0$$

for all multi-indices $\alpha \in \mathbb{N}^d_0$ with $1 \leq |\alpha| < m$,

(c)

$$\mu_\alpha := \int_K x^\alpha \, d\tilde{\lambda}(x)/|\alpha|! \neq 0$$

for all multi-indices $\alpha \in \mathbb{N}^d_0$ with $|\alpha| = m$.

Note that this definition is according to definition 4, if $\tilde{\lambda}$ is represented by an integrable function $f^* \in L^1(K)$.

Theorem 6. Adopt the assumptions of theorem 5 where additionally $\tilde{\lambda}$ has order $m \in \mathbb{N}$. Furthermore assume that $K = \overline{\Omega}$ for an open and bounded subset $\Omega \subset \mathbb{R}^d$ and $f \in C^m_0(\Omega)$. Then there exists a constant $C > 0$ satisfying

$$\|f_{\gamma} - f\|_{\infty} \leq C \gamma^m \|f\|_{C^m(K)} \quad \text{for all} \quad \gamma > 0.$$  

(22)
Proof. As in the proof of theorem 3 we restrict the verification of (22) to \( d = 1 \) and may assume that \( K = [a, b] \) is a closed interval. Again we use a Taylor series expansion up to the order \( m \) to prove the estimate

\[
\sup_{y \in K} \left| \int_a^b (f(\gamma x + y) - f(y)) \, d\bar{\lambda}(x) \right| \leq y^m K_m \| f \|_{C^m(K)},
\]

where

\[
K_m := \int_a^b |x^m| \, d\bar{\lambda}(x),
\]

\( y_{\gamma,x} \in [a, b] \) lies between \( y \) and \( \gamma x + y \) and we used the fact that \( \bar{\lambda} \) has order \( m \). Hence, we showed (22) with \( C := K_m \). \( \square \)

The following lemma stating convergence with rates in the case of noisy data \( g^\delta \) is the \( C(K) \)-analogue of theorem 4.

**Theorem 7.** Let the assumptions of theorems 1 and 6 hold true and \( \nu \) be given as \( 0 < \nu < 1 \). Furthermore assume that \( l_{\gamma} \asymp \gamma^{-k} \) as \( \gamma \to 0 \), \( k > 0 \). If the parameter choice rule \( \gamma(\delta) \) is chosen such that

\[
\gamma(\delta) = O(\delta^{\nu/k}) \quad \text{as} \quad \delta \to 0,
\]

\( f \in \mathcal{C}^m(K) \) and \( \bar{\lambda} \) has order \( m \), where \( m \in \mathbb{N} \) satisfies

\[
m \geq k \frac{1 - \nu}{\nu}, \tag{23}
\]

then

\[
\left\| f_{\gamma(\delta)} - f \right\|_{\infty} \leq c\delta^{1-\nu} \| f \|_{C^m(K)} \quad \text{as} \quad \delta \to 0.
\]

**Proof.** The proof is done as for theorem 4 only by changing the norms accordingly. \( \square \)

5. An application in x-ray diffractometry

We describe briefly an application of the concepts developed in section 4 to the problem of x-ray diffractometry subsuming some results outlined in [21, 23] and adopting them in our investigations.

X-ray diffractometry is a sort of industrial, non-destructive testing and aims for recovering the stress tensor \( \sigma = \sigma_{ij} \) of a specimen from measuring the Bragg angle of reflected x-rays. Mechanical stresses and the elastic strain tensor \( \varepsilon = \varepsilon_{ij} \) are coupled to each other and this connection is expressed by Hooke’s law:

\[
\varepsilon_{ij} = \frac{\nu + 1}{E} \sigma_{ij} - \delta_{ij} \frac{\nu}{E} \sum_{k=1}^3 \sigma_{kk}, \tag{25}
\]
where $\nu$ is the Poisson number and $E$ denotes the modulus of elasticity. In a laboratory system the probe under investigation is rotated by an angle $\varphi$ about the $x_3$-axis and tilted by an angle $\psi$ about the $x_2$-axis transforming the strain tensor to

$$
\varepsilon^L = U^t_{\psi\varphi} \varepsilon U_{\psi\varphi},
$$

where $U_{\psi\varphi} \in SO(3)$ is the matrix that models the tilting and rotating by $\psi$ and $\varphi$, respectively. In x-ray diffraction it is only possible to detect near-surface strains, that is

$$
\varepsilon_{\psi\varphi} := \varepsilon^L_{33}.
$$

Putting (26) into Hooke’s law (25), we get an explicit expression for the connection of $\varepsilon_{\psi\varphi}$ to the stress tensor $\sigma$,

$$
\varepsilon_{\psi\varphi} = \sum_{i,j=1}^{3} \alpha_{ij}(\varphi, \psi) \sigma_{ij},
$$

for certain quantities $\alpha_{ij}(\varphi, \psi)$. Using Bragg’s reflection model for a crystal lattice, an x-ray is reflected only if the beam hits the object under the so-called Bragg angle $\theta$ which is determined by Bragg’s condition

$$
2d \sin \theta = n \lambda,
$$

where $n$ is an integer, $d$ is the distance of the lattice planes and $\lambda$ is the wavelength of the used x-rays. Bragg’s condition implies $dd = -d \cot \theta d\theta$ and thus

$$
\varepsilon_{\psi\varphi} = \frac{dd}{d} \approx -\cot \theta_0 (\theta_{\psi\varphi} - \theta_0),
$$

where $\theta_{\psi\varphi}$ denotes the maximum peak position of the reflected x-ray and $\theta_0$ is the Bragg angle of the unstressed specimen. Stresses cause a shift of the maximum peak position $\theta_{\psi\varphi}$ away from the Bragg angle of the unstressed specimen and by (28) we have a connection of this shift $\theta_{\psi\varphi} - \theta_0$ to the strain $\varepsilon_{\psi\varphi}$ and thus to $\sigma_{ij}$ by (27). Taking into account that the intensity $I(z)$ of the x-ray is attenuated within the specimen according to Lambert–Beer’s law

$$
I(z) = I_0 e^{-\mu z},
$$

where $z$ denotes the penetration depth, $I_0$ is the initial intensity of the outgoing x-rays and $\mu$ is the material specific attenuation coefficient, then we finally have the fundamental connection

$$
-\cot \theta_0 (\theta_{\psi\varphi} - \theta_0) = \sum_{i,j=1}^{3} \alpha_{ij}(\varphi, \psi) \tilde{\sigma}_{ij}(\tau_{\psi}),
$$

where $\tau_{\psi} = \cos \psi \sin \theta_0/(2\mu)$ is the maximal penetration depth of the x-ray depending on the tilt angle $\psi$ and

$$
\tilde{\sigma}_{ij}(\tau) = \frac{1}{\tau} \int_0^{\infty} \sigma_{ij}(z) e^{-z/\tau} dz
$$

denotes the Laplace transform of the stress tensor $\sigma_{ij}$. The problem of x-ray diffraction consists now of reconstructing the stress tensor $\sigma_{ij}$ from measurements of the Bragg angle shifts $\theta_{\psi\varphi} - \theta_0$ to different rotation and the tilt angles $\varphi$ and $\psi$.

Thus, x-ray diffraction contains the problem of inverting the Laplace transform$^3$

$$
L f(\tau) = \tilde{f}(\tau) = \int_0^{\infty} f(z) e^{-\tau z} dz,
$$

$^3$ Note that we omit the factor $1/\tau$, since this factor is not important for our further investigations.
where the penetration depth $\tau$ is only given for finitely many, say $m$ discrete points $\tau = \tau_j$, $j = 1, \ldots, m$. The stresses $\sigma_{ij}$ can be assumed to be smooth, at least continuous, and close to surface, which means we may consider $L$ as a mapping between the Banach spaces $C([\omega_1, \omega_2])$ and $C([\tau_{\min}, \tau_{\max}])$, where $\tau_{\min} := \min\{\tau_j : j = 1, \ldots, m\}$, $\tau_{\max} := \max\{\tau_j : j = 1, \ldots, m\}$. The stresses $\sigma_{ij}$ can be assumed to be smooth, at least continuous, and close to surface, which means we may consider $L$ as a mapping between the Banach spaces $C([\omega_1, \omega_2])$ and $C([\tau_{\min}, \tau_{\max}])$. Since $\exp(-\tau z) \in C([\omega_1, \omega_2] \times [\tau_{\min}, \tau_{\max}])$ the mapping

$$L : C([\omega_1, \omega_2]) \to C([\tau_{\min}, \tau_{\max}])$$

is linear and bounded and thus we are in the situation of section 4. Moreover we may assume that $\omega_1 > 0$ since the translation property of the Laplace transform $L_f(\tau) = e^{\tau L}\{f(\cdot - r)\}(\tau)$ allows us to shift the support of $f$ to any closed interval $[\omega_1 + r, \omega_2 + r]$, $r > 0$.

In [21] the author defines $\bar{e}(z) = \sum_{j=1}^{m} w_j v_j e^{-\tau_j z}$ as a mollifier, where $w_j = \begin{cases} h_j/2, & j = 1 \\ (h_{j-1} + h_j)/2, & 1 < j < m \\ h_{m-1}/2, & j = m \end{cases}$ with $h_j = \tau_{j+1} - \tau_j$ and $v = (v_1, \ldots, v_m)^t \in \mathbb{R}^m$ is chosen such that

$$\int_{\omega_1}^{\omega_2} \bar{e}(z) \, dz = 1.$$ 

One possibility of defining $v_j$ is given by

$$v_j = \frac{1}{m} \left( \int_{\omega_1}^{\omega_2} e^{-\tau_j z} \, dz \right)^{-1}.$$ 

The weights $w_j$ are defined such that $\sum_{j=1}^{m} w_j \sigma_{ij} \exp(-\tau_j z)$ is the trapezoidal sum corresponding to the nodes $\{\tau_j\}$ applied to the integral

$$L^* v(\tau) = \int_{\tau_{\min}}^{\tau_{\max}} v(\tau) e^{-\tau \tau} \, d\tau,$$

which is the adjoint of the Laplace transform applied to a continuous function $v \in C([\tau_{\min}, \tau_{\max}])$ and $v_j = v(\tau_j)$. Via

$$\tilde{\lambda}_E(E) = \int_E \bar{e}(z) \, dz, \quad E \in B([\omega_1, \omega_2]),$$

the mollifier $\tilde{e}$ induces a regular and countable additive measure on the Borel sets of $[\omega_1, \omega_2]$.

In the following we pursue a semi-discrete setting for the solution of $Lf = g$ that takes into account that we measure the data $Lf$ only at the scanning points $\tau = \tau_j$ although we want to recover $f$ in the infinite-dimensional Banach space $C([\omega_1, \omega_2])$. The connection between the discrete data $Lf(\tau_j)$, $j = 1, \ldots, m$, is then established by appropriate interpolation operators mapping $\mathbb{R}^m$ to the Banach space of continuous functions.

We deduce that $\tilde{e}$ has (at least) order 1 and by theorem 5 generates a mollifier $\{e_\gamma : [\omega_1, \omega_2] \to \text{rca}([\omega_1, \omega_2])\}$ for $C([\omega_1, \omega_2])$. The associated reconstruction kernels
\[ \{v_\gamma : [\omega_1, \omega_2] \to rca([\tau_{\min}, \tau_{\max}])\} \text{ are in } [21] \text{ computed with the help of a collocation method postulating that} \]

\[ L^* [v_\gamma (y)](z) = [e_\gamma (y)](z) \]

is satisfied only for finitely many points \( z = z_k, \ k = 1, \ldots, m \). Together with a numerical integration this leads to the system of linear equations

\[ \sum_{j=1}^{m} w_j \tilde{v_\gamma} (y) e^{-\frac{\tau_j}{\tau}} = [e_\gamma (y)](z_k), \quad 1 \leq k, \ j \leq m. \tag{31} \]

The paper presents a condition under which the system (31) is solvable [21]. Having the solution \( \tilde{v_\gamma} (y) \in \mathbb{R}^m \) at hand we set

\[ v_\gamma (y) := I_m \tilde{v_\gamma} (y) \in C([\tau_{\min}, \tau_{\max}]), \tag{32} \]

which denotes the piecewise linear interpolating function and hence is continuous. The following lemma is an immediate consequence of lemma 4.1 from [21].

**Lemma 4.** Let the reconstruction kernel \( \{v_\gamma\} \) for the Laplace transform be determined by (31) and (32). Then there exists a constant \( C_m > 0 \) depending on \( m \) such that

\[ \|v_\gamma (y)\|_{\infty} \leq C_m \gamma^{-1} y^{-1} \quad \text{for all} \quad y \in [\omega_1, \omega_2]. \tag{33} \]

**Proof.** Lemma 4.1 from [21] states that

\[ \|\tilde{v_\gamma} (y)\|_2 \leq \tilde{C}_m \gamma^{-1} y^{-1} \]

for a certain constant \( \tilde{C}_m > 0 \). Since the norms on \( \mathbb{R}^m \) are equivalent, this implies

\[ \|\tilde{v_\gamma} (y)\|_{\infty} \leq C_m \gamma^{-1} y^{-1} \]

with another constant \( C_m > 0 \). Together with

\[ \|v_\gamma (y)\|_{\infty} = \|I_m \tilde{v_\gamma} (y)\|_{\infty} \leq \|\tilde{v_\gamma} (y)\|_{\infty}, \]

this proves (33). \( \square \)

Of course as a continuous function \( v_\gamma (y) \) induces a measure in \( rca([\tau_{\min}, \tau_{\max}]) \) whose total variation can be estimated by the sup-norm.

**Lemma 5.** We have that

\[ \|v_\gamma (y)\|_B \leq (\tau_{\max} - \tau_{\min}) C_m \gamma^{-1} y^{-1} \quad \text{for all} \quad y \in [\omega_1, \omega_2]. \tag{34} \]

**Proof.** Estimate (34) follows immediately from (33) and

\[ \|v_\gamma (y)\|_B = \sup_{\|f\|_1 \leq 1} \left| \int_{\tau_{\min}}^{\tau_{\max}} f(x) [v_\gamma (y)](x) \, dx \right| \leq (\tau_{\max} - \tau_{\min}) \|v_\gamma (y)\|_{\infty}. \quad \square \]

Finally we are able to estimate \( l_\gamma \), the sup-norm of \( \|v_\gamma (y)\|_\infty \).

**Lemma 6.** From (34) we deduce that

\[ l_\gamma \leq c_m \gamma^{-1} \quad \text{for all} \quad \gamma > 0, \tag{35} \]

that is \( l_\gamma = O(\gamma^{-1}) \).
Proof. The proof is a simple consequence of (34) and $y^{-1} \leq \omega^{-1}$. We have that $c_m = (\tau_{\text{max}} - \tau_{\text{min}})C_m$.

We are now able to prove convergence rates for the method of approximate inverse applied to the Laplace transform. To this end we denote the approximate inverse of $L f$ by

$$f^\delta_\gamma(y) := \langle v_\gamma(y), g^\delta_\gamma \rangle_{rca}([\tau_{\text{min}}, \tau_{\text{max}}]) \times C([\tau_{\text{min}}, \tau_{\text{max}}])$$

where $g^\delta_\gamma \in C([\tau_{\text{min}}, \tau_{\text{max}}])$ represents noise-contaminated measure data. Indeed applying piecewise linear interpolation to the noise-contaminated discrete measure data $g^\delta_m$ results in a continuous function $g^\delta_\gamma$. We note that in this case the mapping $y \mapsto v_\gamma(y)$ is continuous what follows from (31) and the continuity of $y \mapsto e_\delta(y)$ and $I_m$. From lemma 3 we deduce then that $f^\delta_\gamma \in C([\omega_1, \omega_2])$ and the approximate inverse is a bounded operator, since

$$\sup_{y \in [\omega_1, \omega_2]} \|v_\gamma(y)\|_B < \infty.$$

Theorem 8. Let $g = L f$ for $f \in C([\omega_1, \omega_2])$ be the exact measure data, which are given only at $m \in \mathbb{N}$ points

$$g(\tau_j) = (g_m)_j, \quad j = 1, \ldots, m,$$

and assume that only noise-contaminated data $g^\delta_m \in \mathbb{R}^m$ are available satisfying $\|g - I_m g^\delta_m\|_\infty < \delta$, where $I_m : \mathbb{R}^m \to C([\tau_{\text{min}}, \tau_{\text{max}}])$ denotes the piecewise linear interpolation operator. Furthermore assume that the mollifier $\hat{e}$ is given by (30) and that the associated reconstruction kernel is calculated via (31) and (32).

If the parameter choice rule $\gamma = \gamma(\delta)$ is chosen such that

$$\gamma(\delta) \asymp \delta^{1/2} \quad \text{as} \quad \delta \to 0,$$

then there exists $c > 0$ with

$$\|f^\delta_{\gamma(\delta)} - f\|_\infty \leq c\delta^{1/2} \|f\|_{C^1([\omega_1, \omega_2])} \quad \text{as} \quad \delta \to 0$$

provided that $f \in C^1([\omega_1, \omega_2])$.

Proof. We apply theorem 7 to prove the convergence statements. From (35) we see that $k = 1$ in theorem 7. For given $v$ with $0 < v < 1$, we have that due to (23) and $m = 1$,

$$1 \geq \frac{1}{v}.$$

Hence, the highest possible convergence order is achieved when $1 = (1 - v)/v \Leftrightarrow v = 1/2$. But for $v = 1/2$, we immediately get (36) from (24).

Remark 4. If we assume that in (30) we could $v = (v_1, \ldots, v_m)^t$ define such that $\hat{e}$ has the largest possible order $m$, then following the lines in the proof of theorem 8 we would get a convergence rate $O(\delta^{2m})$. In that case the number of scanning points would determine the optimal convergence rate in x-ray diffractometry, which fits to the results of Plato and Vainikko [14], where they investigated the discretization of some regularization methods applied to general operator equations and proved that the optimal convergence rates are affected by the discretization step size. Moreover considering $m \to \infty$ we obtain the overall, in an asymptotic sense, optimal rate $O(\delta)$. Of course also in the semi-discrete setting this asymptotic rate $O(\delta)$ can never be achieved in practice but is only important from a theoretical point of view.
6. Conclusions

We were able to extend the method of the approximate inverse \( f_\gamma = A_\gamma A f \) to a general Banach space setting for solving operator equations where the operator \( A \) is injective. We showed that the optimal convergence rate would be \( O(\delta) \) and could prove the convergence rates \( O(\delta^{1-\nu}) \) for any \( 0 < \nu < 1 \) only if the mollifier allows for an approximation \( \| f_\gamma(\delta) - f \|_X = O(\delta^{1-\nu}) \). In the case of \( L^p \)-spaces with \( 1 \leq p < \infty \) and of spaces of continuous functions on a compact set \( C(K) \) we proved that these convergence rates can be accomplished whenever the mollifier has an order that is large enough. Finally we showed the practical relevance of our investigations by verifying the convergence results for x-ray diffractionmetry. Here the optimal convergence rate is determined not only by the order of the mollifier but also by the number of data scanning points.

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