4. Exercise Sheet to the Numerical Internship in Computerized Tomography

Shannon’s sampling theorem and the normal equation

Exercise 1: Shannon’s sampling theorem
For \( g \in L^1(\mathbb{R}) \) let
\[
    f(t) := \sum_{j=1}^{N} \alpha_j (g * \text{sinc})(\pi b_j(t - t_j))
\]
with \( t, t_j, \alpha_j, b_j, b \in \mathbb{R} \) and \( 0 < b_j \leq b \) for \( j = 1, \ldots, N \).

(a) Compute \( \hat{f} \) and show that \( f \) is bandlimited.

(b) How do you choose \( h > 0 \) such that \( f \) can be reconstructed from the values \( f(h \cdot k), k \in \mathbb{Z} \)?
    What is the minimum size of the details contained in \( f \)?

Exercise 2: Normal equation
Let \( A : X \rightarrow Y \) be a continuous linear operator between Hilbert spaces \( X \) and \( Y \). Show that for \( x^* \in X \) and \( y \in Y \) the following statements are equivalent:

(a) \( \|Ax^* - y\| \leq \|Ax - y\| \) for all \( x \in X \),

(b) \( A^*Ax^* = A^*y \).

Hint: For “(a) \( \Rightarrow \) (b)” and arbitrary \( u \in X \) consider the mapping
\[
    f_u(t) := \|A(x^* + tu) - y\|^2, \quad t \in \mathbb{R}.
\]

Exercise 3: Solution set of the normal equation
Let \( A : X \rightarrow Y \) be a continuous linear operator between Hilbert spaces \( X \) and \( Y \). For \( y \in \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \) let
\[
    L(y) := \{x \in X : A^*Ax = A^*y\}
\]
be the solution set of the normal equation. Show:

(a) \( L(y) \) is closed and convex.

(b) For \( x^\dagger \in L(y) \) we have
\[
    \|x^\dagger\|_X \leq \|x\|_X \quad \text{for all} \quad x \in L \Leftrightarrow x^\dagger \in \mathcal{N}(A)^\perp.
\]