Development of Algorithms in Computerized Tomography

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This paper is dedicated to Hermann and Dr. Charlotte Deutsch.

ABSTRACT. In this paper we present a general approach to derive inversion algorithms for tomographic applications, the so-called approximate inverse. Three different techniques for calculating a reconstruction kernel are shown and applied to invert the Radon transform, to compute approximations in the limited angle problem and to solve the 3D cone beam reconstruction problem. Reconstructions from real data in this case are presented.

1. Introduction

In this paper we present some principles in designing inversion algorithms in tomography. We concentrate on linear problems arising in connection with the Radon and the x-ray transform. In the original 2D x-ray CT problem the Radon transform served as mathematical model. Here one integrates over lines and the problem is to recover a function from its line integrals. The same holds in the 3D x-ray case, but in 3D the Radon transform integrates over planes, in general over $N-1$ - dimensional hyper planes in $\mathbb{R}^N$. Hence here the so-called x-ray transform is the mathematical model. Further differences are in the parametrization of the lines. The 3D - Radon transform merely appears as tool to derive inversion formula. In the early days of MRI (magnetic resonance imaging), at those days called NMR, nuclear magnetic resonance, it served as a mathematical model, see for example Marr-Chen-Lauterbur [**MCL81**], but then, due to the limitations of computer power in those days one changed the measuring procedure and scanned the Fourier transform of the searched-for function in two dimensions. Nowadays the Radon transform reappeared, now in three and even four dimensions as mathematical model in EPRI (electron parametric resonance imaging) where spectral - spatial information is the goal, see e.g. Kuppusamy et al [**KCSWZ95**].

The paper is organized as following. We start with a general principle for reconstruction information from measured data, the so-called approximate inverse, see Louis [**Lou96**], Louis-Maass [**LM90**]. The well-known inversion of the Radon

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transform is considered a model case for inversion. The singular value decomposition is then used to compute reconstruction kernels for the limited angle problem. Finally we consider a 3D x-ray problem and present reconstructions from real data.

2. Approximate Inverse as a Tool for Deriving Inversion Algorithms

The integral operators appearing in medical imaging are typically compact operators between suitable Hilbert spaces. The inverse operator of those compact operators with infinite dimensional range are not continuous, which means that the unavoidable data errors are amplified in the solution. Hence one has to be very careful in designing inversion algorithms. They have to balance the demand for highest possible accuracy and the necessary damping of the influence of the unavoidable data errors. From the theoretical point of view, exact inversion formulae are nice, but they do not take care of data errors. The way out of this dilemma is the use of approximate inversion formulas whose principles are explained in the following.

For approximating the solution of

$$Af = g$$

we apply the method of approximate inverse, see Louis [**Lou96**]. The basic idea works as follows: choose a so-called mollifier $e_\gamma(x,y)$ which, for a fixed reconstruction point x is a function of the variable y and which approximates the delta distribution for the point x. The parameter γ acts as regularization parameter. Simply think in the case of one spatial variable x of

$$e_\gamma(x,y) = \frac{1}{2\gamma}\chi_{[x-\gamma,x+\gamma]}(y)$$

where χ_Ω denotes the characteristic function of Ω. Then the mollifier fulfills

$$\int e_\gamma(x,y)dy = 1 \tag{2.1}$$

for all x and the function

$$f_\gamma(x) = \int f(y)e_\gamma(x,y)dy$$

converges for $\gamma \to 0$ to f. The larger the parameter γ the larger is the interval where the averaging takes place and hence the stronger is the smoothing. Now solve for fixed reconstruction point x the auxiliary problem

$$A^*\psi_\gamma(x,\cdot) = e_\gamma(x,\cdot) \tag{2.2}$$

where $e_\gamma(x,\cdot)$ is the chosen approximation to the delta distribution for the point x, and put

$$\begin{aligned} f_\gamma(x) &= \langle f, e_\gamma(x,\cdot)\rangle \\ &= \langle f, A^*\psi_\gamma(x,\cdot)\rangle = \langle Af, \psi_\gamma(x,\cdot)\rangle = \langle g, \psi_\gamma(x,\cdot)\rangle \\ &=: S_\gamma g(x)\,. \end{aligned}$$

The operator S_γ is called the approximate inverse and ψ_γ is the reconstruction kernel. To be precise it is the approximate inverse for approximating the solution f of $Af = g$. If we choose instead of e_γ fulfilling (2.1) a wavelet, then f_γ can be interpreted as a wavelet transform of f. Wavelet transforms are known to approximate in a certain sense derivatives of the transformed function f, see [**LMR97**].

Hence this is a possibility to find jumps in f as used in contour reconstructions, see [**LM93**].
The advantage of this method is that ψ_γ can be pre-computed independently of the data. Furthermore, invariances and symmetries of the operator A^* can directly be transformed into corresponding properties of S_γ as the following consideration shows, see Louis [**Lou96**]. Let T_1 and T_2 be two operators intertwining with A^*

$$A^* T_2 = T_1 A^* .$$

If we choose a standard mollifier E and solve $A^*\Psi = E$ then the solution of Eq. (2.2) for the special mollifier $e_\gamma = T_1 E$ is given as

$$\psi_\gamma = T_2 \Psi .$$

As an example we mention, that if A^* is translation invariant; i.e., $T_1 f(x) = T_2 f(x) = f(x-a)$, then also the reconstruction kernel is translation invariant.

Sometimes it is easier to cheque these conditions for A itself. Using $AT_1^* = T_2^* A$ we get the above relations by using the adjoint operators.

This method is presented in [**Lou99**] as general regularization scheme to solve inverse problems. Generalizations are also given. The application to vector fields is derived by Schuster [**Sch00**].

If the auxiliary problem is not solvable then its minimum norm solution leads to the minimum norm solution of the original problem.

There are several possibilities for solving (2.2). Besides the straight forward approach we can easily solve this equation if the inverse operator is of the form

$$A^{-1} = A^* B .$$

The solution of (2.2) is then given as

$$\psi_\gamma = B A e_\gamma \tag{2.3}$$

as the following equations show

$$A^* \psi_\gamma = e_\gamma = A^{-1} A e_\gamma = A^* B A e_\gamma .$$

The next situation we consider is that a singular system of the operator is known. Assume the operator being compact as mapping from a Hilbert space X into a Hilbert space Y. Then the operator has a complete singular system consisting of normalized functions $v_n \in X$, $u_n \in Y$ and nonnegative numbers σ_n such that

$$A v_n = \sigma_n u_n \, , A^* u_n = \sigma_n v_n .$$

Information on singular value decomposition in connection with the Radon transform can be found in [**Lou89**], [**Nat86**]. The solution of (2.2) can then be calculated as

$$\psi_\gamma(x,y) = \sum_{n=0}^{\infty} \sigma_n^{-1} \langle e_\gamma(x,\cdot), v_n \rangle u_n(y) \tag{2.4}$$

Finally we want to mention the case, where the inverse is again given as

$$A^{-1} = A^* B$$

and we want to save the structure of this formula using the operator A^*. Here we start by first smoothing the data with a smoothing operator $\tilde{M}_\gamma$ and then we invert, see [**Lou99**] Theorems 6 and 7. The approximate inverse has then the form

$$S_\gamma = A^{-1}\tilde{M}_\gamma\,.$$

We pre-compute a reconstruction kernel in the following way. Let $w_\gamma(y,\cdot)$ be an approximation of the delta distribution on the data for the point y. Then we put

$$\phi_\gamma(y,\cdot) = Bw_\gamma(y,\cdot) \tag{2.5}$$

and compute

$$f_\gamma = A^* B_\gamma g$$

with

$$B_\gamma g(y) = \langle g, \phi_\gamma(y,\cdot)\rangle\,.$$

3. Inversion of the Radon Transform

We apply the above approach to derive inversion algorithms for the Radon transform. The Radon transform in $\mathbb{R}^N$ is defined as

$$\mathbf{R}f(\theta,s) = \int\limits_{\mathbb{R}^N} f(x)\delta(s - x^\top\theta)\,dx$$

for unit vectors $\theta \in S^{N-1}$ and $s \in \mathbb{R}$. Its inverse is

$$\mathbf{R}^{-1} = c_N \mathbf{R}^* I^{1-N} \tag{3.1}$$

where $\mathbf{R}^*$ is the adjoint operator from L_2 to L_2, also called the backprojection, defined as

$$\mathbf{R}g(x) = \int_{S^{N-1}} g(\theta, x^\top\theta)d\theta\,,$$

I^α is the Riesz potential defined via the Fourier transform as

$$\widehat{(I^\alpha g)}(\xi) = |\xi|^{-\alpha}\widehat{g(\xi)},$$

acting on the second variable of $\mathbf{R}f$ and the constant

$$c_N = \frac{1}{2}(2\pi)^{1-N}\,.$$

see e.g. [**Nat86**]. We start with a mollifier $e_\gamma(x,\cdot)$ for the reconstruction point x and get

$$\begin{aligned}\mathbf{R}^*\psi_\gamma(x,\cdot) &= e_\gamma(x,\cdot)\\ &= c_N\mathbf{R}^* I^{1-N}\mathbf{R}e_\gamma(x,\cdot)\end{aligned}$$

leading to

$$\psi_\gamma(x;\theta,s) = c_N I^{1-N}\mathbf{R}e_\gamma(x;\theta,s)\,.$$

The Radon transform for fixed θ is translational invariant; i.e., if we denote by $\mathbf{R}_\theta f(s) = \mathbf{R}f(\theta,s)$, then

$$\mathbf{R}_\theta T_1^a f = T_2^{a^\top\theta}\mathbf{R}_\theta f$$

with the shift operators $T_1^a f(x) = f(x-a)$ and $T_2^t g(s) = g(s-t)$. If we chose a mollifier $\bar{e}_\gamma$ supported in the unit ball centred around 0 that is shifted to x as

$$e_\gamma(x,y) = 2^{-N}\bar{e}_\gamma(\frac{x-y}{2})$$

then also e_γ is supported in the unit ball and the reconstruction kernel fulfills

$$\psi_\gamma(x;\theta,s) = \frac{1}{2}\bar{\psi}_\gamma(\theta, \frac{s - x^\top\theta}{2})$$

as follows from the general theory in [**Lou96**] and as was used for the 2D case in [**LS96**].
Furthermore, the Radon transform is invariant under rotations; i.e.,

$$\mathbf{R}T_1^U = T_2^U\mathbf{R}$$

for the rotation $T_1^U f(x) = f(Ux)$ with unitary U and ${T_2}^U g(\theta,s) = g(U\theta,s)$. If the mollifier is invariant under rotation; i.e.,

$$\bar{e}_\gamma(x) = \bar{e}_\gamma(\|x\|)$$

then the reconstruction kernel is independent of θ leading to the following observation.

THEOREM 3.1. *Let the mollifier $e_\gamma(x,y)$ be of the form*

$$e_\gamma(x,y) = 2^{-N}\bar{e}_\gamma(\|x-y\|/2)$$

then the reconstruction kernel is a function only of the variable s and the algorithms is of filtered backprojection type

$$f_\gamma(x) = \int_{S^{n-1}}\int_{\mathbb{R}} \psi_\gamma(x^\top\theta - s)\mathbf{R}f(\theta,s)dsd\theta\,. \tag{3.2}$$

We described here the approach mentioned in (2.3). First references to this technique can be found in the work of Grünbaum [**DG81**] and Solmon, [**HS88**].

4. The Filtered Backprojection for the Radon Transform in 2 and 3 Dimensions

In the following we describe the derivation of the filtered backprojection, see Theorem 3.1, for two and three dimensions. As seen in Formula (3.1) the inverse operator of the Radon transform in $\mathbb{R}^N$ has the representation

$$\mathbf{R}^{-1} = \mathbf{R}^*B$$

with

$$B = c_N I^{1-N}.$$

Hence we can apply Formula (2.2) for deriving reconstruction kernels. They then can be represented as

$$\psi_\gamma = c_N I^{1-N}\mathbf{R}e_\gamma. \tag{4.1}$$

As mollifier we choose a translational and rotational invariant function

$$\bar{e}_\gamma(x,y) = e_\gamma(\|x-y\|)$$

whose Radon transform then is a function of the variable s only. Taking the Fourier transform of Equation (4.1) we get

$$\begin{aligned}\hat{\psi}_\gamma(\sigma) &= c_N(\widehat{I^{1-N}(\mathbf{R}e_\gamma)})(\sigma)\\ &= \frac{1}{2}(2\pi)^{(1-N)/2}|\sigma|^{N-1}\hat{e}_\gamma(\sigma),\end{aligned}$$

where in the last step we have used the projection theorem

$$\hat{f}(\sigma\theta) = (2\pi)^{(1-N)/2}\widehat{\mathbf{R}_\theta f}(\sigma).$$

So, we can proceed in the following two ways. Either we prescribe the mollifier e_γ, where the Fourier transform is then computed to

$$\hat{e}_\gamma(\sigma) = \sigma^{1-N/2} \int_0^\infty e_\gamma(s) s^{N/2} J_{N/2-1}(s\sigma) ds$$

where J_ν denotes the Bessel function of order ν. On the other hand we prescribe

$$\hat{e}_\gamma(\sigma) = (2\pi)^{-N/2} F_\gamma(\sigma)$$

with a suitably chosen filter F_γ leading to

$$\hat{\psi}_\gamma(\sigma) = \frac{1}{2}(2\pi)^{1/2-N} |\sigma|^{N-1} F_\gamma(\sigma).$$

If F_γ is the ideal low-pass; i.e., $F_\gamma(\sigma) = 1$ for $|\sigma| \leq \gamma$ and 0 otherwise, then the mollifier is easily computed as

$$e_\gamma(x, y) = (2\pi)^{-N/2} \gamma^N \frac{J_{N/2}(\gamma \|x - y\|)}{(\gamma \|x - y\|)^{N/2}} .$$

In the two-dimensional case the calculation of ψ leads to the so called RAM-LAK filter, which has the disadvantage to produce ringing artefacts due to the discontinuity in the Fourier domain.

More popular for 2D is the filter

$$F_\gamma(\sigma) = \begin{cases} \operatorname{sinc} \frac{\sigma\pi}{2\gamma} & , \quad |\sigma| \leq \gamma, \\ 0 & , \quad |\sigma| > \gamma \end{cases}$$

From this we compute the kernel ψ_γ by inverse Fourier transform to get for $\gamma = \pi/h$ where h is the stepsize on the detector; i.e., $h = 1/q$ if we use $2q + 1$ points on the interval $[-1, 1]$ and $s = s_\ell = \ell h$, $\ell = -q, \ldots, q$

$$\psi_\gamma(s_\ell) = \frac{\gamma^2}{\pi^4} \frac{1}{1 - 4\ell^2} ,$$

known as Shepp - Logan kernel.

The algorithm of filtered backprojection is a stable discretization of the above described method using the composite trapezoidal rule for computing the discrete convolution. Instead of calculating the convolution for all points $\theta^\top x$ the convolution is evaluated for equidistant points ℓh and then a linear interpolation is applied. Nearest neighbour interpolation is not sufficiently accurate, higher order interpolation is not bringing any improvement because the interpolated functions are not smooth enough. Then the composite trapezoidal rule is used for approximating the backprojection. Here one integrates a periodic function, hence, as shown with the Euler- Maclaurin summation formula, this formula is highly accurate. The filtered backprojection then consists of two steps. Let the data $\mathbf{R}f(\theta, s)$ be given for the directions $\theta_j = (\cos\varphi_j, \sin\varphi_j)$, $\varphi_j = \pi(j-1)/p$, $j = 1, ..., p$ and the values $s_k = kh$, $h = 1/q$ and $k = -q, ..., q$.

Step 1: For j=1,...,p, evaluate the discrete convolutions

$$v_{j,\ell} = h \sum_{k=-q}^{q} \psi_\gamma(s_\ell - s_k) \mathbf{R}f(\theta_j, s_k), \;\; \ell = -q, ..., q. \tag{4.2}$$

Step 2: For each reconstruction point x compute the discrete backprojection

$$\tilde{f}(x) = \frac{2\pi}{p} \sum_{j=1}^{p} ((1-\eta) v_{j,\ell} + \eta v_{j,\ell+1} \tag{4.3}$$

where, for each x and j, ℓ and η are determined by

$$s = \theta_j^\top x, \ell \leq s/h < \ell + 1, \eta = s/h - \ell$$

see e.g. [**Nat86**].

In the three - dimensional case we can use the fact, that the operator I^{-2} is local,

$$I^{-2} g(\theta, s) = \frac{\partial^2}{\partial s^2} g(\theta, s)$$

If we want to keep this local structure in the discretization we choose

$$F_\gamma(\sigma) = 2(1 - \cos(h\sigma))/(h\sigma)^2$$

leading to

$$\psi_\gamma(s) = (\delta_\gamma - 2\delta_0 + \delta_{-\gamma})(s) \tag{4.4}$$

Hence, the application of this reconstruction kernel is nothing but the central difference quotient for approximating the second derivative. The corresponding mollifier then is

$$e_\gamma(y) = \begin{cases} (2\pi)^{-1} h^{-2} |y|^{-1} & , \text{ for} |y| < h, \\ 0 & , \text{ otherwise,} \end{cases}$$

see [**Lou83**]. The algorithm has the same structure as mentioned above for the 2D case.

In order to get reconstruction formulas for the fan beam geometry coordinate transforms can be used, the structure of the algorithms does not change.

5. The Reconstruction Kernel for the Limited Angle Transform

As an example for the use of the singular value decomposition for computing the reconstruction kernel we consider the so-called limited angle problem. For practical applications see the paper of Quinto, [**Qui06**] in this volume. Singular value decompositions are known for several of these integral transforms appearing in tomography. For the classical 2D Radon transform the early result of Cormack [**Cor64**] can be interpreted as singular value decomposition. For the Radon transform in arbitrary dimensions it is given in [**Lou84**], for the x-ray transform in arbitrary dimensions and the parallel geometry it was derived by Maass, [**Maa87**]. Singular value decompositions for the fan-beam geometry are simple modifications. In the 2D case it has been used by Louis-Rieder [**LR89**] for deriving algorithms for region-of-interest tomography. Quinto [**Qui88**] used a singular value decomposition for developing algorithms for the exterior problem.

The singular value decomposition can also be used to answer the question of resolution and the practically invisible objects. An approach which gives more insight is based on wavefront sets, see Quinto [**Qui93**]. In the following we want to use the svd for computing a reconstruction kernel, rather than using them directly on the data to compute approximations in the missing range as done in Louis [**Lou80**].

We first describe the limited angle problem and formulate its singular value decomposition, originally presented in Louis [**Lou86**], now using the notion introduced by Slepian [**Sle78**].
We consider the parallel geometry of x-ray CT where the mathematical model is the *Radon transform.* Let f be an L_2-function of compact support in the unit disk $\Omega \subset \mathbb{R}^2$, after a possible rescaling. Denote by $\theta \in S^1$ the unit vector

$$\theta = \theta(\varphi) = (\cos\varphi, \sin\varphi)^\top$$

and by $\theta^\perp = \theta(\varphi + \pi/2) = (-\sin\varphi, \cos\varphi)^\top$ the vector orthogonal to θ.
The Radon transform is defined as

$$\begin{aligned} \mathbf{R}f(\theta, s) &= \int_{\mathbb{R}^2} f(x)\delta(s - x^\top\theta)\, dx \\ &= \int_{\mathbb{R}} f(s\theta + t\theta^\perp)\, dt\,. \end{aligned}$$

We assume $\mathbf{R}f(\theta, \cdot)$ to be given for all θ with

$$\theta \in S_\phi := \{\theta(\varphi) : |\varphi| \le \phi\} \cup \{\theta(\varphi) : |\varphi - \pi| \le \phi\} \subset S^1$$

with $0 < \phi < \frac{\pi}{2}$. Note that in [**Lou86**] this was the missing range, hence the results change accordingly. Due to the symmetry of the Radon transform it suffices to know the data on one of the subsets of S_ϕ.

With the truncated cylinder

$$Z_\phi = S_\phi \times [-1, 1]$$

and the weight

$$w(s) = (1 - s^2)^{1/2}$$

we consider

$$\mathbf{R}_\phi : L_2(\Omega) \to L_2(Z_\phi, w^{-1})$$

where the scalar product in the latter space is defined as

$$< f, g >_{w^{-1}} = \int_{-\Phi}^{\Phi} \int_{-1}^{1} w^{-1}(s) f(\varphi, s) g(\varphi, s) ds d\varphi.$$

In a first step we present the singular value decomposition of the limited angle transform $\mathbf{R}_\phi$. Note that $\mathbf{R}_{\pi/2} = \mathbf{R}$.
We make use of the following notion, introduced by Slepian [**Sle78**].
Let the $N \times N$ matrix $\rho(N, \phi/\pi)$ be given as

$$\rho(N, \phi/\pi)_{mn} = \frac{\sin 2\phi(m-n)}{\pi(m-n)}\,, \quad m, n = 0, \ldots, N-1\,.$$

with the diagonal elements $2\phi/\pi$.

Let $\lambda_k(N, \phi/\pi)$, $k = 0, \ldots, N-1$ be its eigenvalues and the

$$v^{(k)}(N, \phi/\pi) \in \mathbb{R}^N,\ k = 0, \ldots, N-1$$

its with respect to the Euclidean norm normalized eigenvectors. They are sections of the discrete prolate spheroidal sequences and are related to *the discrete spheroidal wave functions* $u_k(N, \phi/\pi, \varphi/\pi)$ via

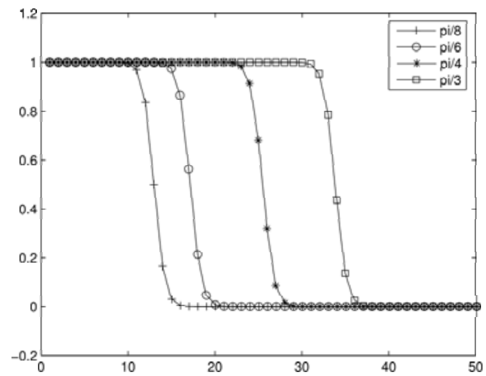

FIGURE 1. Eigenvalues of the Slepian matrix for different values of Φ.

$$u_k(N,W;f) = \varepsilon_k \sum_{n=0}^{N-1} v_n^{(k)}(N,W)\, e^{-\imath\pi\,(N-1-2n)\,f}\,, \quad k = 0,\dots,N-1\,, \tag{5.1}$$

where

$$\varepsilon_k = \begin{cases} 1 & , \quad k \quad \text{even,} \\ \imath & , \quad k \quad \text{odd,} \end{cases}$$

see [**Sle78**, Formula (26)].
Finally we denote by U_m the *Chebychev polynomials* of the second kind and by

$$Q_{ml}(r) = \Big(2(m+1)\Big)^{1/2} r^l\, P^{(0,l)}_{(m-l)/2}(2r^2-1)$$

the normalized *Zernike polynomials* with $P_k^{(\alpha,\beta)}$ the Jacobi polynomials.

THEOREM 5.1. *Let* $\mathbf{R}_\phi : L_2(\Omega) \to L_2(Z_\phi, w^{-1})$. *Then* $(f^\phi_{ml}, g^\phi_{ml}; \sigma^\phi_{ml})$, $m \geq 0$, $0 \leq l \leq m$ *with*

$$\begin{aligned} f^\phi_{ml}(r\omega(\vartheta)) &= (2\pi)^{-1/2} \sum_{\mu=0}^{m} v_\mu^{(l)}(m+1,\phi/\pi)\, Q_{m,|2\mu-m|}(r)\, e^{\imath(2\mu-m)\,\vartheta}\,, \\ g^\phi_{ml}(\theta(\varphi),s) &= \frac{1}{\pi}\, w(s)\, U_m(s)\, \varepsilon_l^{-1}\, u_l(m+1,\phi/\pi,\varphi/\pi)\lambda_l(m+1,\phi/\pi)^{-1/2} \\ \sigma^\phi_{ml} &= 2\Big(\frac{\pi}{m+1}\, \lambda_l(m+1,\phi/\pi)\Big)^{1/2} \end{aligned}$$

form a complete singular system for the limited angle problem.

PROOF. The proof follows [**Lou86**] and is sketched here. Using the singular value decomposition for the full range case, see e.g. [**Lou84**], [**Nat86**]

$$\begin{aligned} f_{ml}(r\omega(\vartheta)) &= (2\pi)^{-1/2}\, Q_{m,|2l-m|}(r)\, e^{\imath(2l-m)\vartheta}\,, \\ g_{ml}(\theta(\varphi),s) &= \frac{1}{\pi}\, w(s)\, U_m(s)\, e^{\imath(2l-m)\varphi}\,, \\ \sigma_{ml}=\sigma_m &= 2\Big(\frac{\pi}{m+1}\Big)^{1/2} \end{aligned}$$

it remains to orthogonalize on the truncated cylinder.
The f^{ϕ}_{ml} are a linear combination of the f_{ml},

$$f^{\phi}_{ml} = \sum_{\mu=0}^{m} v_\mu(\ell)(m+1,\Phi/\pi) f_{m\mu}$$

and hence they are complete on $L_2(\Omega)$ due to the completeness and orthogonality of the functions $f_{m\ell}$ and the orthonormality of the vectors $v^{(\ell)}(m+1,\Phi/pi), \ell = 1,\dots,m$. Then we expand $\mathbf{R}^*\mathbf{R}f^{\phi}_{ml}$ in terms of these functions. The Fourier coefficients are

$$\begin{aligned} c_{nk} &= \langle \mathbf{R}^*_\phi \mathbf{R}_\phi f^{\phi}_{ml}, f^{\phi}_{nk}\rangle_{L_2(\Omega)} \\ &= \langle \mathbf{R}_\phi f^{\phi}_{ml}, \mathbf{R}_\phi f^{\phi}_{nk}\rangle_{L_2(Z_\phi, w^{-1})}\,. \end{aligned}$$

Using $\mathbf{R}f_{ml} = \sigma_m g_{ml}$ we get

$$\begin{aligned} c_{nk} &= \frac{4}{\pi}\frac{1}{\sqrt{(m+1)(n+1)}} \underbrace{\int_{-1}^{1} w(s)\, U_m(s)\, U_n(s)\, ds}_{=(\pi/2)\delta_{mn}} \times \\ &\quad \times \sum_{\mu=0}^{m}\sum_{\nu=0}^{n} v^{(l)}_\mu(m+1,\phi/\pi)\, v^{(k)}_\nu(n+1,\phi/\pi) \underbrace{\int_{S_\phi} e^{\imath(2\mu-m+n-2\nu)\varphi}\, d\varphi}_{=2\pi\rho(m+1,\phi/\pi)_{\mu\nu}} \\ &= \frac{4\pi}{m+1}\, \delta_{mn}\, v^{(l)}(m+1,\phi/\pi)^\top \rho(m+1,\phi/\pi)\, v^{(k)}(m+1,\phi/\pi) \\ &= \frac{4\pi}{m+1}\, \lambda_l(m+1,\phi/\pi)\, \delta_{mn}\, \delta_{kl}\,, \end{aligned}$$

which means that

$$\mathbf{R}^*_\phi \mathbf{R}_\phi f^{\phi}_{ml} = \frac{4\pi}{m+1}\, \lambda_l(m+1,\phi/\pi)\, f^{\phi}_{ml} = (\sigma^{\phi}_{ml})^2\, f^{\phi}_{ml}\,.$$

Using $R_\phi f_{ml} = \sigma^{\phi}_{ml} g^{\phi}_{ml}$ and relation (5.1) complete the proof.

□

The eigenvalues $\lambda_l(m+1,\phi/\pi)$ show the well known behaviour that they are close to 1, if $l < (m+1)\phi/\pi$ and the rest is close to zero, if they are ordered from large to small. Their exponential decay, see [**Sle78**], means that the problem is severely ill – posed . As a consequence some of the components of the solution, namely those belonging to small singular values, are practically invisible. For a more geometrical interpretation see Qunito's approach with the wave front sets, [**Qui06**]. This also affects the computation of the reconstruction kernel, see the discussion after Theorem5.2.

The aim is now to represent an approximation f_γ of the equation $\mathbf{R}_\phi f = g$ in the form

$$f_\gamma(x) = \langle g, \tilde{\psi}_\gamma^\phi \rangle_{L_2(Z,w^{-1})} .$$

To this end we solve, according to (2.4)

$$\mathbf{R}_\phi^* \tilde{\psi}_\gamma^\phi = e_\gamma$$

for a prescribed mollifier resulting in $f_\gamma = \langle f, e_\gamma \rangle$.
The limited angle transform intertwines with two shift operators, namely

$$T_1^x f(y) = f\Big(x - y\Big)$$

and

$$T_2^t g(\omega, s) = g\Big(\omega, s - t\Big)$$

as

$$\mathbf{R}_\Phi T_1^x = T_2^{x^\top \omega} \mathbf{R}_\Phi .$$

This means that, if we choose a shift invariant mollifier $e_\gamma(x, y) = T_1^x E_\gamma(y)$ with a standard mollifier E_γ, then the reconstruction kernel $\tilde{\psi}_\gamma(x; \omega, s)$ is of the form

$$\tilde{\psi}_\gamma(x; \omega, s) = \bar{\psi}_\gamma(\omega, s - x^\top \omega) ,$$

see [**Lou96**].
This means that it suffices to compute the reconstruction kernel for one point, preferably $x = 0$, and than shift the kernel.
Using the singular value decomposition of the limited angle transform we can represent the minimum norm solution of

$$\mathbf{R}_\phi^* \bar{\psi}_\gamma = E_\gamma$$

as

$$\bar{\psi}_\gamma(\omega, s) = \sum_{m=0}^{\infty} \sum_{l=0}^{m} \frac{1}{\sigma_{ml}^\phi} \langle E_\gamma, f_{ml}^\phi \rangle_{L_2(\Omega)} \, g_{ml}^\phi(\omega, s) . \tag{5.2}$$

Using polar coordinates the scalar products are computed as

$$\begin{aligned} \langle E_\gamma, f_{ml}^\phi \rangle_{L_2(\Omega)} &= (2\pi)^{-1/2} \sum_{\mu=0}^{m} v_\mu^{(l)}(m+1, \phi/\pi) \times \\ &\times \int_0^1 r \, Q_{m,|2\mu-m|}(r) \int_0^{2\pi} E_\gamma(r\,\omega(\vartheta)) \, e^{\imath(2\mu-m)\vartheta} \, d\vartheta \, dr . \end{aligned}$$

If the mollifier is radially symmetric, i.e.

$$E_\gamma(y) = \bar{E}_\gamma(|y|) ,$$

then we get

$$\begin{aligned}\langle E_\gamma, f^\phi_{ml}\rangle_{L_2(\Omega)} &= (2\pi)^{-1/2}\sum_{\mu=0}^{m} v^{(l)}_\mu(m+1,\phi/\pi)\times\\ &\quad\times\int_0^1 r\,Q_{m,|2\mu-m|}(r)\,\bar E_\gamma(r)\,dr\cdot 2\pi\,\delta_{\mu,m/2}\\ &= (2\pi)^{1/2}\,v^{(l)}_{m/2}(m+1,\phi/\pi)\int_0^1 r\,Q_{m,0}(r)\,\bar E_\gamma(r)\,dr\end{aligned}$$

for m even, and 0 otherwise.
Using the symmetry relation of the discrete spheroidal sequence

$$v^{(l)}_n(N,W) = (-1)^l\, v^{(l)}_{N-1-n}(N,W)\,,$$

see [**Sle78**, Formula (23)], we observe

$$v^{(l)}_{m/2}(m+1,\phi/\pi) = 0 \quad \text{for } l \text{ odd}.$$

This means that both m and l have to be even.
From (5.2) then follows

$$\bar\psi_\gamma(\omega,s) = \sum_{\substack{m=0\\ m\,even}}^{\infty}\ \sum_{\substack{l=0\\ l\,even}}^{m} \frac{1}{\sigma^\phi_{ml}}\,\langle E_\gamma,\, f^\phi_{ml}\rangle_{L_2(\Omega)}\, g^\phi_{ml}(\omega,s)\,.$$

Denoting by

$$\alpha_{m,\gamma} := \int_0^1 r\,Q_{m,0}(r)\,\bar E_\gamma(r)\,dr$$

and using that g^ϕ_{ml} contains the factor $w(s)$, the scalar product in $L_2(Z_\phi, w^{-1})$ we can state the result avoiding weighted scalar products in the following way.

THEOREM 5.2. *The reconstruction kernel ψ^ϕ_γ for the limited angle problem has the form*

$$\begin{aligned}\tilde\psi^\phi_\gamma(\omega,s) &= \frac{1}{\pi\sqrt2}\sum_{m=0}^{\infty}(2m+1)^{1/2}\,U_{2m}(s)\,\alpha_{2m,\gamma}\times\\ &\quad\times\sum_{l=0}^{m} u_{2l}(2m+1,\phi/\pi,\varphi/\pi)\,\lambda_{2l}(2m+1,\phi/\pi)^{-1}\end{aligned}$$

leading to the representation of the solution of $\mathbf{R}_\phi f = g$ as

$$f_\gamma(x) = \int_{-\phi}^{\phi}\int_{-1}^{1}\tilde\psi^\phi_\gamma(\omega(\varphi), s - x^\top\omega)\,g(\omega(\varphi),s)\,ds\,d\varphi\,.$$

The numerical problem in the evaluation of ψ^ϕ_γ is based on the decay of the eigenvalues $\lambda_l(m+1,\phi/\pi)$. The series has to be truncated early enough not to produce numerical instabilities. But this restricts the resolution in the reconstruction too much, which we avoid by computing the reconstruction kernel as a correction

term for the full range kernel.
Using

$$\psi_\gamma(s) = \frac{1}{\pi\sqrt{2}} \sum_{m=0}^{\infty} (2m+1)^{1/2} U_{2m}(s)\, \alpha_{2m,\gamma}\,,$$

we get the following result:

THEOREM 5.3. *The reconstruction kernel for the limited angle problem has the following, numerically more attractive, form:*

$$\tilde{\psi}_\gamma^\phi(\omega, s) = \psi_\gamma(s) + \tilde{\psi}_\gamma^{\phi,C}(\omega, s)$$

with

$$\begin{aligned} \tilde{\psi}_\gamma^{\phi,C}(\omega, s) &= \frac{1}{\pi\sqrt{2}} \sum_{m=0}^{\infty} (2m+1)^{1/2} U_{2m}(s)\, \alpha_{2m,\gamma} \times \\ & \Big(\sum_{l=0}^{m} u_{2l}(2m+1, \phi/\pi; \varphi/\pi)\, \lambda_{2l}(2m+1, \phi/\pi)^{-1} - 1 \Big). \end{aligned}$$

REMARK 5.4. The following symmetry conditions hold

$$\psi_\gamma^\phi(\omega(-\varphi), s) = \psi_\gamma^\phi(\omega(\varphi), s)\,, \psi_\gamma^{\phi,c}(\omega(-\varphi), s) = \psi_\gamma^{\phi,c}(\omega(\varphi), s)\,,$$
$$\psi_\gamma^\phi(\omega, -s) = \psi_\gamma^\phi(\omega, s)\,, \psi_\gamma^{\phi,c}(\omega, -s) = \psi_\gamma^{\phi,c}(\omega, s)\,.$$

For practical computations we cut-off the representation of the correction term in the above theorem.

6. Inversion Formula for the 3D Cone Beam Transform

In the following we consider the X-ray reconstruction problem in three dimensions when the data is measured by firing an X-ray tube emitting rays to a 2D detector. The movement of the combination source - detector determines the different scanning geometries. In many real - world applications the source is moved on a circle around the object. From a mathematical point of view this has the disadvantage that the data are incomplete, the condition of Tuy-Kirillov is not fulfilled. This condition says, that essentially the data are complete for the three - dimensional Radon transform. All planes through a point x have to cut the scanning curve Γ. We base our considerations on the assumptions that this condition is fulfilled, the reconstruction from real data nevertheless is then from the above described circular scanning geometry, because other data is not available to us so far.

A first theoretical presentation of the reconstruction kernel was given by Finch [**Fin87**], invariances were then used in the group of the author to speed-up the computation time considerably, so that real data could be handled, see [**Lou03**]. See also the often used algorithm from Feldkamp et al. [**FDK84**] and the contribution of Defrise and Clack [**DC94**]. The approach of Katsevich [**Kat02**] differs from our approach that he avoids the Crofton symbol by restricting the backprojection to a range dependent on the reconstruction point x.

The presentation follows Louis [**Lou04**].

The mathematical model here is the so-called X-ray transform, where we denote with $a \in \Gamma$ the source position, where $\Gamma \subset \mathbb{R}^3$ is a curve, $\theta \in S^2$ is the direction of the ray:

$$\mathbf{D}f(a,\theta) = \int_0^\infty f(a+t\theta)dt$$

The adjoint operator of D as mapping from $L_2(\mathbb{R}^3) \longrightarrow L_2(\Gamma \times S^2)$ is given as

$$\mathbf{D}^* g(x) = \int_\Gamma |x-a|^{-2} g\left(a, \frac{x-a}{|x-a|}\right) da$$

Most attempts to find inversion formulae are based on a relation between X-ray transform and the 3D Radon transform, the so-called *Formula of Grangeat*, first published in Grangeat's PhD thesis [**Gr87**], see also [**Gr91**] :

$$\frac{\partial}{\partial s}\mathbf{R}f(\omega, a^\top\omega) = -\int_{S^2} \mathbf{D}f(a,\theta)\delta'(\theta^\top\omega)d\theta.$$

PROOF. We copy the proof from [**NW01**]). It consists of the following two steps. i)$\int_{\mathbb{R}} \mathbf{R}f(\omega,s)\psi(s)ds = \int_{\mathbb{R}^3} f(x)\psi(x^\top\omega)dx$
ii) $\int_{S^2} \mathbf{D}f(a,\theta)h(\theta)d\theta = \int_{\mathbb{R}^3} f(x)h(\frac{x-a}{|x-a|})|x-a|^{-2}dx$
Putting $\psi(s) = \delta'(s-a^\top\omega)$ and use $h(\theta) = \delta'(\theta^\top\omega)$ and the fact that δ' is homogeneous of degree -2 in $\mathbb{R}^3$ completes the proof. $\square$

We note the following rules for δ':

i)

$$\int_{S^2} \psi(a^\top\omega)\delta'(\theta^\top\omega)d\omega = -a^\top\theta \int_{S^2\cap\theta^\perp} \psi'(a^\top\omega)d\omega$$

ii)

$$\int_{S^2} \psi(\omega)\delta'(\theta^\top\omega)d\omega = -\int_{S^2\cap\theta^\perp} \frac{\partial}{\partial\theta}\psi(\omega)d\omega$$

Starting point is now the inversion formula for the 3D Radon transform

$$f(x) = -\frac{1}{8\pi^2}\int_{S^2} \frac{\partial^2}{\partial s^2}\mathbf{R}f(\omega, x^\top\omega)d\omega \tag{6.1}$$

rewritten as

$$f(x) = \frac{1}{8\pi^2}\int_{S^2}\int_{\mathbb{R}} \frac{\partial}{\partial s}\mathbf{R}f(\omega,s)\delta'(s-x^\top\omega)dsd\omega$$

We assume in the following that the Tuy-Kirillov condition is fulfilled. Then we can change the variables as: $s = a^\top\omega$, n is the Crofton symbol; i.e., the number of source points $a \in \Gamma$ such that $a^\top\omega = x^\top\omega$, $m = 1/n$ and get

$$\begin{aligned}
f(x) &= \frac{1}{8\pi^2}\int_{S^2}\int_\Gamma (\mathbf{R}f)'(\omega, a^\top\omega)\delta'((a-x)^\top\omega)|a'^\top\omega|m(\omega,a^\top\omega)dad\omega \\
&= -\frac{1}{8\pi^2}\int_{S^2}\int_\Gamma\int_{S^2} \mathbf{D}f(a,\theta)\delta'(\theta^\top\omega)d\theta\delta'((a-x)^\top\omega)|a'^\top\omega|m(\omega,a^\top\omega)dad\omega \\
&= -\frac{1}{8\pi^2}\int_\Gamma |x-a|^{-2}\int_{S^2}\int_{S^2} \mathbf{D}f(a,\theta)\delta'(\theta^\top\omega)d\theta\delta'(\frac{(x-a)}{|x-a|}^\top\omega) \\
&\quad \times|a'^\top\omega|m(\omega,a^\top\omega)dad\omega
\end{aligned}$$

where we again used that δ' is homogeneous of degree -2. We now introduce the following operators

$$T_1 g(\omega) = \int_{S^2} g(\theta)\delta'(\theta^\top \omega)d\theta \tag{6.2}$$

and we use T_1 acting on the second variable as

$$T_{1,a} g(\omega) = T_1 g(a, \omega)\,.$$

We also use the multiplication operator

$$M_{\Gamma,a} h(\omega) = |a'^\top \omega| m(\omega, a^\top \omega) h(\omega)\,. \tag{6.3}$$

and state the following result.

THEOREM 6.1. *Let the condition of Tuy-Kirillov be fulfilled. Then the inversion formula for the cone beam transform is given as*

$$f = -\frac{1}{8\pi^2}\mathbf{D}^* T_1 M_{\Gamma,a} T_1 \mathbf{D} f \tag{6.4}$$

with the adjoint operator $\mathbf{D}^*$ *of the cone beam transform and* T_1 *and* $M_{\Gamma,a}$ *as defined above.*

Note that the operators $\mathbf{D}^*$ and M depend on the scanning curve Γ.

This form allows for computing reconstruction kernels. To this end we have to solve the equation

$$D^* \psi_\gamma = e_\gamma$$

in order to write the solution of $\mathbf{D} f = g$ as

$$f(x) = < g, \psi_\gamma(x, \cdot) > \quad .$$

In the case of exact inversion formula e_γ is the delta distribution, in the case of the approximate inversion formula it is an approximation of this distribution, see the method of approximate inverse. Using that $\mathbf{D}^{-1} = -\frac{1}{8\pi^2}\mathbf{D}^* T_1 M_{\Gamma,a} T_1$ we get

$$\mathbf{D}^* \psi = \delta = -\frac{1}{8\pi^2}\mathbf{D}^* T_1 M_{\Gamma,a} T_1 \mathbf{D}\delta$$

and hence

$$\psi = -\frac{1}{8\pi^2} T_1 M_{\Gamma,a} T_1 \mathbf{D}\delta \tag{6.5}$$

We can explicitly give the form of the operators T_1 and $T_2 = MT_1$. The index at ∇ indicates the variable with respect to which the differentiation is performed.

$$\begin{aligned} T_1 g(a,\omega) &= \int_{S^2} g(a,\theta)\delta'(\theta^\top \omega)d\theta \\ &= -\omega^\top \int_{S^2 \cap \omega^\perp} \nabla_2 g(a,\theta) d\theta \end{aligned}$$

and

$$\begin{aligned} T_1 M_{\Gamma,a} h(a,\alpha) &= \int_{S^2} \delta'(\omega^\top \alpha)|a'^\top \omega| m(\omega, a^\top \omega) h(a,\omega) d\omega \\ &= -a'^\top \alpha \int_{S^2 \cap \alpha^\perp} \operatorname{sign}(a'^\top \omega) m(\omega, a^\top \omega) h(a,\omega) d\omega \\ &\quad -\alpha^\top \int_{S^2 \cap \alpha^\perp} |a'^\top \alpha| \nabla_1 m(\omega, a^\top \omega) h(a,\omega) d\omega \\ &\quad -a^\top \alpha \int_{S^2 \cap \alpha^\perp} |a'^\top \omega| \nabla_2 m(a, a^\top \omega) h(a,\omega) d\omega \\ &\quad - \int_{S^2 \cap \alpha^\perp} |a'^\top \omega| m(\omega, a^\top \omega) \frac{\partial}{\partial \alpha} h(a,\omega) d\omega \end{aligned}$$

Note that the function m is piecewise constant, the derivatives are then Delta - distributions at the discontinuities with factor equal to the height of the jump; i.e., 1/2.

Depending on the scanning curve Γ invariances have to be used. For the circular scanning geometry this leads to similar results as mentioned in [**Lou03**]. In the following we present a reconstruction from data provided by the Fraunhofer Institut for Nondestructive Testing (IzfP) in Saarbrücken. The detector size was $(204.8mm)^2$ with 512^2 pixels and 400 source positions on a circle around the object. The second data set was provided by the Deusches Krebsforschungszentrum (DKFZ) Heidelberg, with the same number of data, namely 10.4 million. The mollifier used is

$$e_\gamma(y) = (2\pi)^{-3/2}\gamma^{-3} \exp\Big(-\frac{1}{2}\Big|\frac{y}{\gamma}\Big|^2\Big).$$

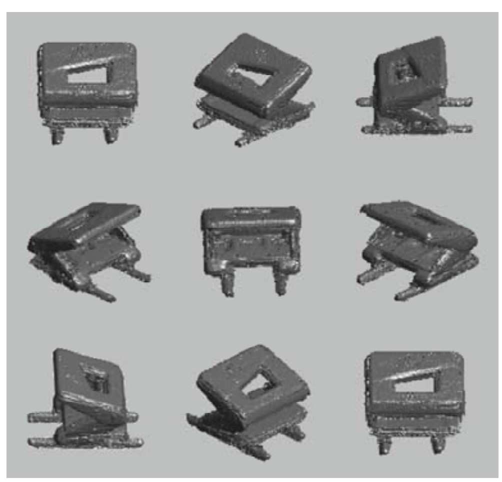

FIGURE 2. Reconstruction of a perforator.

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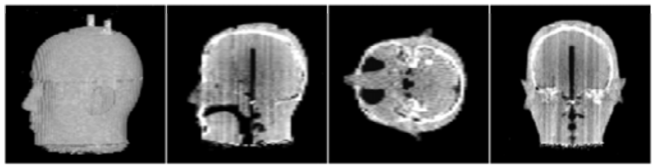

FIGURE 3. A human skull filled with plastic, and with skin, ears, nose etc. modelled with plastic. The bars which fix this skull are visible.

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