Computing Reconstruction Kernels for circular 3D Cone Beam Tomography

A. K. Louis, T. Weber and D. Theis

Institute for Applied Mathematics, Saarland University, P.O. Box 15 11 50, 66041 Saarbrücken, Germany

E-mail: louis@num.uni-sb.de / weber@num.uni-sb.de / theis@num.uni-sb.de

Goals

We want to reconstruct scalar values (densities) from x-ray measurements, using a cone of x-rays, measuring the loss of intensity with a plane detector array. The process looks like the following:

- Send x-rays through object
- Measure loss of intensity
- Reconstruct the density
- Mathematically: invert the cone beam transform

3D CT has applications in medicine and non-destructive testing.

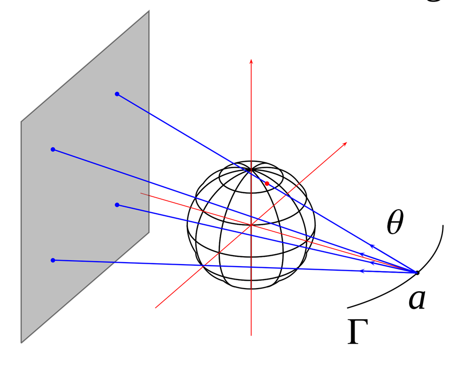

Figure 1: Measurement setup

Cone beam transform D

It is defined by

$$[\mathbf{D}f](a,\theta) = \int_0^\infty f(a+t\theta)\,dt, \qquad (1)$$

- a source point on the scanning curve Γ,
- θ direction of ray,
- f the searched for density,

and the adjoint operator is

$$[\mathbf{D}^* g](x) = \int_\Gamma \frac{1}{\|x-a\|^2}\, g\left(\frac{x-a}{\|x-a\|}\right) da. \qquad (2)$$

Approximate Inverse

- We are looking for the solution f of $Af = g$.
- Instead of f, we reconstruct an approximate version f_γ with the property

$$f_\gamma \to f \quad \text{as } \gamma \searrow 0.$$

- Solve auxiliary problem

$$A^*\psi_\gamma(x,\cdot) = e_\gamma(x,\cdot), \qquad (3)$$

with chosen *mollifier* e_γ and *reconstruction kernel* ψ_γ and define

$$f_\gamma(x) = \left\langle f, e_\gamma(x,\cdot)\right\rangle = \langle g, \psi(x,\cdot)\rangle\,. \qquad (4)$$

- Used for both scalar [2] and vector [5] tomography.
- Advantages
  - Auxiliary problem solvable independent of the measurement.
  - Invariances of the operators can be taken into account.
  - Using approximate invariances, the memory requirements can be kept down[4].
  - The mollifier can be adapted to the problem. In our case, it should approximate the delta distribution, centered at x.

Inversion Formula for the 3D Cone Beam Transform[3]

Define operators

$$[T_1 g](\omega) = \int_{S^2} g(\theta)\,\delta'(\langle\theta,\omega\rangle)\,d\theta, \qquad (5)$$

$$[M_\Gamma h](a,\omega) = |\langle\dot a,\omega\rangle|\; m(\omega,\langle a,\omega\rangle)\; h(\omega), \qquad (6)$$

with $m = 1/n$, n the Crofton symbol and we get

$$f = \frac{1}{8\pi^2}\mathbf{D}^* T_1 M_\Gamma T_1 \mathbf{D} f, \qquad (7)$$

$$\psi_\gamma = \frac{1}{8\pi^2} T_1 M_\Gamma T_1 \mathbf{D} e_\gamma. \qquad (8)$$

Explicit formula for the reconstruction kernel.

Starting point for the above inversion formula was the *formula of Grangeat*[1]

$$\frac{\partial}{\partial s}\mathbf{R}f(\omega,s)\Big|_{s=\langle a,\omega\rangle} = -\int_{S^2}\mathbf{D}f(a,\theta)\delta'(\langle\theta,\omega\rangle)\,d\theta. \qquad (9)$$

Computing the reconstruction kernel

Gaussian as mollifier

$$e_\gamma(x,y) = (2\pi)^{-3/2}\frac{1}{\gamma^3}\, e^{-\frac{\|x-y\|^2}{2\gamma^2}}, \qquad (10)$$

$$T_1\mathbf{D}e_\gamma(a,\omega,x) = \frac{(2\pi)^{-1/2}}{\gamma^3}\, e^{-\frac{1}{2\gamma^2}\langle a-x,\omega\rangle^2}\,\langle a-x,\omega\rangle\,. \qquad (11)$$

Set $n \equiv 2$, hence $m = 1/2$.

Analytical formula for reconstruction kernel

$$\psi_\gamma(a,\theta,x) = -\frac{C}{2\pi}\Big[\frac{p_3}{p_4}\big\{\langle\dot a,\theta\rangle - 2\alpha\,\langle a-x,\theta\rangle\, p_3\big\} \times \int_0^1 e^{p_1\left[p_2 t^2-1\right]}\,dt + p_4\,\langle a-x,\theta\rangle\, e^{p_1\left[p_2-1\right]}\Big], \qquad (12)$$

$$\alpha := \frac{1}{2\gamma^2}, \qquad C := (2\pi)^{-3/2}\frac{1}{\gamma^3},$$

$$p_1 := \alpha\,\|a-x-\langle a-x,\theta\rangle\,\theta\|^2\,,$$

$$p_2 := \frac{\langle a-x-\langle a-x,\theta\rangle\,\theta, \dot a-\langle\dot a,\theta\rangle\,\theta\rangle^2}{\|\dot a-\langle\dot a,\theta\rangle\,\theta\|^2\,\|a-x-\langle a-x,\theta\rangle\,\theta\|^2},$$

$$p_3 := \langle a-x-\langle a-x,\theta\rangle\,\theta, \dot a-\langle\dot a,\theta\rangle\,\theta\rangle\,,$$

$$p_4 := \|\dot a-\langle\dot a,\theta\rangle\,\theta\|\,.$$

If θ lies parallel to $(x-a)$, this simplifies to

$$\psi_\gamma(a,\theta,x) = -\frac{C}{2\pi}\,\|\dot a-\langle\dot a,\theta\rangle\,\theta\|^2\,\langle a-x,\theta\rangle\,. \qquad (13)$$

Numerical results

Kernel

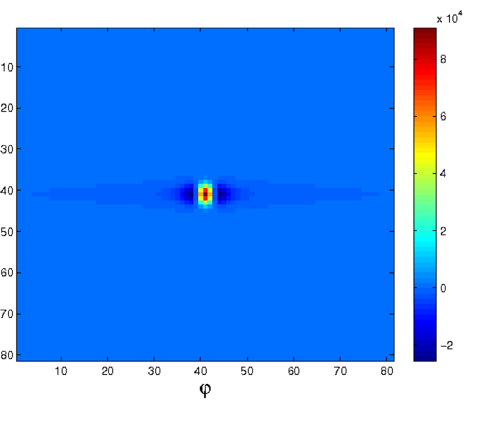

Figure 2: Reconstruction kernel with $\gamma = 0.01$ and 513^2 elements.

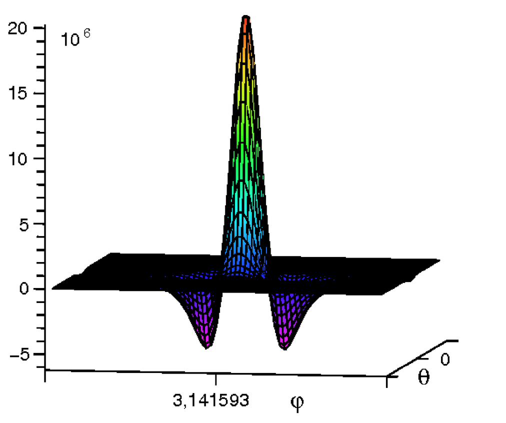

Figure 3: Reconstruction kernel with $\gamma = 0.00165$.

Reconstructions from real data

Data were kindly provided by Fraunhofer IzfP, (http://www.izfp.fhg.de/).

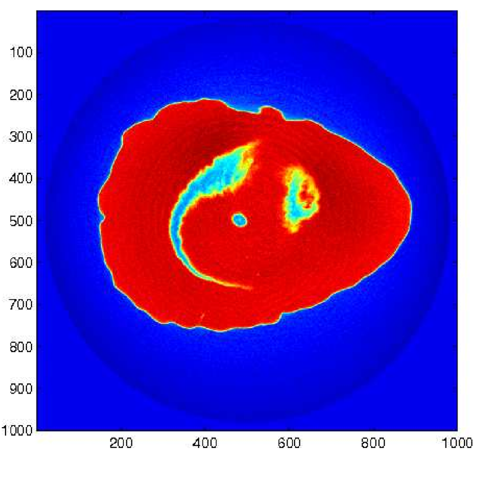

Figure 4: Reconstruction at height $z = -0.22$.

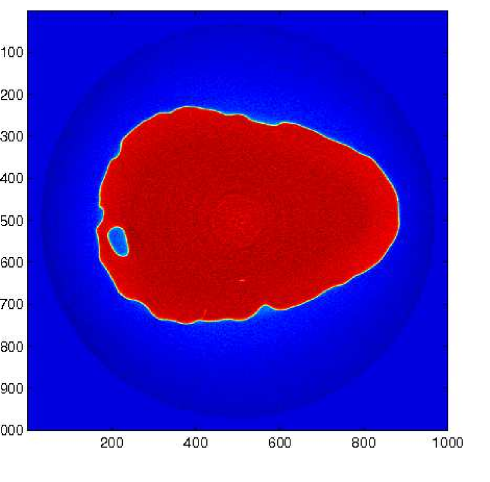

Figure 5: Reconstruction at height $z = -0.28$.

Measurement parameters	
Detector array	1024 × 1024
Projections	800
Source – Detector	~ 126 cm
Source – Object	~ 5.1 cm
Reconstruction parameters	
Reconstruction grid	1000 × 1000
γ	0.00165

Figure 6: Original bust/sculpture

Measurement parameters	
Detector array	2048 × 2048
Projections	400
Source – Detector	~ 126 cm
Source – Object	~ 2.9 cm
Reconstruction parameters	
Reconstruction grid	3000 × 3000
γ	0.00174

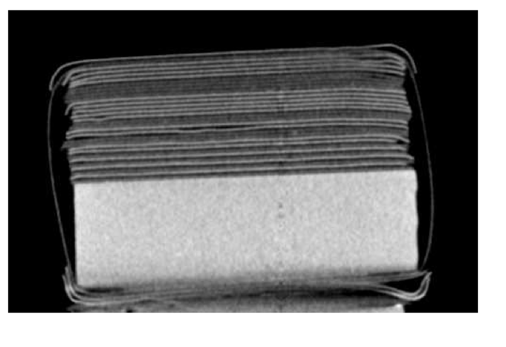

Figure 7: Test object, consisting of sheets of aluminium and adhesive, sliced at $x = 0$.

References

[1] Pierre Grangeat. *Analyse d'un Système d'Imagerie 3D par Reconstruction à partir de Radiographics X en Géométrie Conique*. PhD thesis, Ecole Nationale Supérieure des Télécommunications, 1987.

[2] A K Louis. Approximate inverse for linear and some nonlinear problems. *Inverse Problems*, 12(2):175–190, 1996.

[3] A K Louis. Development of algorithms in computerized tomography. In Gestur Olafsson and Eric Todd Quinto, editors, *The Radon Transform, Inverse Problems, and Tomography (Proceedings of Symposia in Applied Mathematics)*, volume 63 of *Proceedings of Symposia in Applied Mathematics*, pages 25–42, Boston, MA, USA, 2006. AMS.

[4] Alfred K Louis. Filter design in three-dimensional cone beam tomography: circular scanning geometry. *Inverse Problems*, 19(6):S31–S40, 2003.

[5] Thomas Schuster. The 3d doppler transform: elementary properties and computation of reconstruction kernels. *Inverse Problems*, 16(3):701–722, 2000.